Finite Size Effects in Integrable Quantum Field Theories

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Abstract

The study of Finite Size Effects in Quantum Field Theory allows the extraction of precious perturbative and non-perturbative information. The use of scaling functions can connect the particle content (scattering theory formulation) of a QFT to its ultraviolet Conformal Field Theory content. If the model is integrable, a method of investigation through a nonlinear integral equation equivalent to Bethe Ansatz and deducible from a light-cone lattice regularization is available. It allows to reconstruct the S-matrix and to understand the locality properties in terms of Bethe root configurations, thanks to the link to ultraviolet CFT guaranteed by the exact determination of scaling function. This method is illustrated in practice for Sine-Gordon / massive Thirring models, clarifying their locality structure and the issues of equivalence between the two models. By restriction of the Sine-Gordon model it is also possible to control the scaling functions of minimal models perturbed by $\Phi_{(1,3)}$.

1 Introduction

Finite Size Effects (FSE) are widely recognized to play an important role in modern Statistical Mechanics and Quantum Field Theory (QFT). From a statistical point of view, it is known that no phase transitions take place in a finite volume system. For example, the specific heat $c(T)$, that is divergent at the critical point in infinite volume, looses it divergence if the system has finite size; one observes only a rounded peak in the plot of $c(T)$ versus T (see fig.1). Moreover, there is only an interval around critical temperature T_c where the FSE are relevant. Out of this interval, they are negligible (because only near T_c the correlation length is comparable with the size of the system). The interesting fact is that specific heat (and other critical quantities) have a scaling behaviour (i.e. varying the size L) that is fixed by the (infinite size) critical exponents (see [1]). This is a general fact: as argued in [2], the UV behaviour of the scaling functions (see later) is fixed by the conformal dimensions of the operators that belong to the universality class of the critical point (i.e. the CFT describing the critical point of the statistical system).

Also in QFT interesting phenomena appear. If the space-time geometry is a cylinder of circumference L, Casimir effects change the energy of a two body interaction, because particles interact in the two possible directions, as shown in fig. 2. New radiative corrections to the self-energy of a propagating particle may appear because of the closed geometry. For example, a massive particle propagating in a finite 1+1 dimensional space-time, with

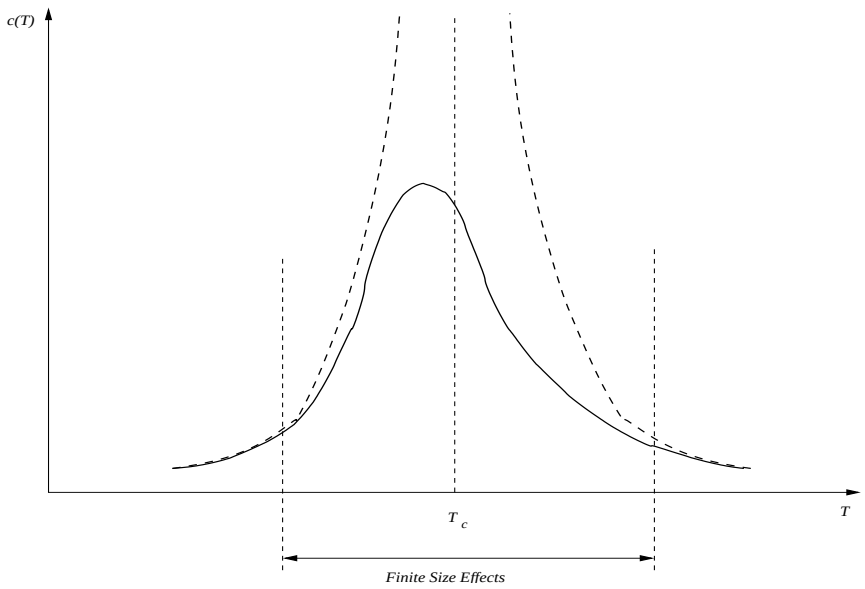


Figure 1: Specific Heat near a phase transition point in infinite volume (dashed line) and in finite volume (solid line)

periodic boundary conditions in the space direction and an infinitely extended time direction will have its propagator renormalized by the contributions of radiative corrections, as depicted in the second cylinder of fig.2. While on an infinite space only local virtual emissions are allowed, on a cylindrical geometry one can conceive virtual emissions that travel around the whole cylinder before coming back to the bare particle. Of course they are exponentially depressed if compared to the traditional local emissions, therefore, if the cylinder is large, their contributions to the self-energy of the propagator is negligible. However, if the cylinder circumference L is small enough, there is room left by Heisenberg principle for such virtual object to travel around the world and come back from the other side, thus giving new contributions that modify the self-energy.

Lüscher [3, 4] has estimated the relevance of such a phenomenon by computing the corrections to the mass of the particle due to FSE. For a massive self-interacting boson with ϕ^4 potential in 3+1 dimensions, they are represented by

$$\Delta M(L) = -\frac{1}{16\pi^2}\frac{3}{mL}\int dy\, e^{-\sqrt{m^2+y^2}}F(iy) + O(e^{-\sqrt{\frac{3}{2}}mL})$$

m is the physical mass of the particle in infinite volume; F is the analytic continuation of the forward elastic scattering amplitude S with incoming momenta p_1, p_2 and outcoming momenta p_3, p_4

$$F(\nu) = S(p_1, p_2|p_3, p_4)$$

where $\nu = \frac{\omega(p_1)\omega(p_2)-p_1\cdot p_2}{m}$ and $\omega(p) = \sqrt{m^2+\vec{p}^2}$. Saddle point integration peaks a contri-

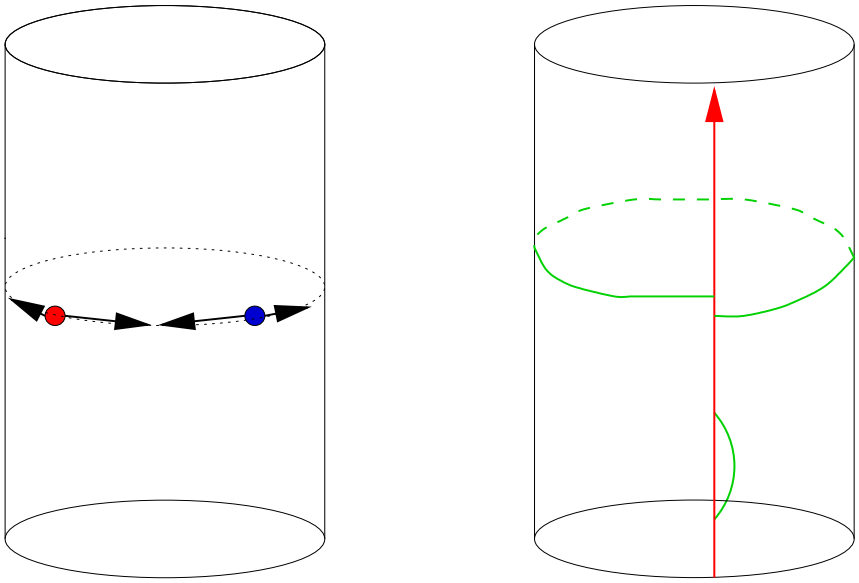


Figure 2: Radiative corrections to the self energy of a particle in finite volume

bution from around $y = 0$, so that the result is

$$\Delta M \propto \frac{e^{-mL}}{L\sqrt{mL}} + ...$$

A very useful application of this estimate shows up in lattice QFT calculations. For example consider a 4 dimensional lattice, of lattice spacing a, on which you want to perform Montecarlo simulations of QCD. The lattice sizes are $T \times L \times L \times L$, with, say, $L = 20a$. In lattice QFT, the discretization of space is to be intended as a regulator, *i.e.* it is equivalent to the introduction of a momentum cutoff $\Lambda_{cut} \approx \frac{1}{a}$ which must be much larger than the typical hadron masses, if we want to get reliable data from the lattice. For example, we can choose $a < 0.1$ fm (1 fm $= 10^{-13}$ cm); then $L = 20a = 2$ fm, which is comparable with the proton radius (Compton wavelength). The proton in such a lattice is "squeezed", which means that the FSE are relevant. One way to overcome this difficulty is to build larger and larger computers, which, as history of the last 20 years indicates, goes to introduce other sorts of difficulties of financial, technical and also principal nature. The other possibility, at the cost of a pencil, is to be able to compute the FSE scaling of quantities related to the proton, from which we can extract physical data even from such small lattices. Of course nobody is nowadays able to estimate with accuracy the L dependence of quantities in a formidable problem like QCD, but progresses have been made in two dimensional QFT's, some of them sharing interesting phenomena with QCD or other interesting realistic 4 dimensional QFT's. In particular, there are methods to exactly calculate the L dependence of physical quantities in the class of *Integrable* two dimensional QFT. This will be the main topic of the present review.

Let us continue to analyze this lattice QFT example. Finite Size with periodic boundary conditions implies quantization of momenta in units of order $\Delta p = \frac{2\pi}{L} \geq 600$ MeV. Pions

in such a lattice ($m_\pi = 139$ MeV) have virtual processes strongly affected by FSE. This illustrates a bit more numerically the observation about virtual processes made above.

This discussion leads us to some important observations:

- As momenta are quantized, the energy spectrum for finite L is discrete
- FSE do not depend strongly on the lattice spacing a, they depend on the scale $L \gg a$ and become irrelevant when $L \gg$ Compton wavelength of the lightest particle (of mass m). Therefore the physical quantities may be expressed, up to overall factors encoding their dimensionality, as dimensionless functions of the dimensionless scale $l = mL$. To such objects the usual denomination of *scaling functions* is given.
- As $L \gg a$, FSE are relevant to renormalized QFT. l can play the role of dimensionless scale parameter of a sort of Renormalization Group flow as we go to illustrate by the following heuristic arguments:
  - the limit $l \to 0$ means or $L \to 0$ i.e. the cylinder shrinks to a line, and there is no spatial dimension anymore, which is an uninteresting unphysical situation, or $m \to 0$, which means that the lightest particle mass becomes irrelevant compared to the energies in play in the system, which is exactly the setup of the ultraviolet (UV) regime. Therefore, the limit $l \to 0$ reproduces the UV regime of the model.
  - the limit $l \to \infty$ means or $m \to \infty$, i.e. the situation where the masses of the particles are much greater than the scattering transferred momenta, which defines the infrared (IR) region, or $L \to \infty$, i.e. the reconstruction of the QFT on the infinite volume. Note that these two limits are not in contradiction: actually the theory on the whole infinite volume is described in terms of the S-matrix, which is an intrinsically IR object, in the sense that it describes the scattering of asymptotic particles without looking at the details of the interaction at short range.
- Both the UV and IR limits thus identified are fixed points of the QFT where the Callan - Symanzyk beta function annullates. They are therefore scale invariant points. Scale invariance in relativistic QFT often implies conformal invariance. Therefore the UV and IR points of the flow obtained by varying l from 0 to ∞ are to be identified with Conformal Field Theories (CFT).
  - In the case of massive QFT, the IR CFT is trivial. The CFT description is useful in this case only at the UV point, where the theory can be seen as a relevant perturbation of a given CFT. At IR instead, the most effective description is in terms of the S-matrix.
  - In the case of massless (but not scale invariant) theories, both the UV and IR points are nontrivial CFT's, and the scaling functions interpolate between the physical quantities of these two CFT's.

2 Excursus on the Renormalization Group in 1+1 dimensions

In this section we quickly recall the main properties of a two-dimensional QFT under scale transformations. In general a QFT is not invariant under scale transformations, due to the presence of dimensionful parameters, appearing already at the classical level or generated by the renormalization procedure. To investigate this phenomenon it is useful to think in terms of the so called *(Wilson) Space of Actions* [5, 6]. Let us suppose that the action $S = \int d^D x\, \mathcal{L}$ of a theory depends on a set of fundamental fields and their derivatives, and on a set of dimensionless coupling constants $\vec{g} \equiv (g_1, ..., g_n)$. The variation of the Lagrangian density $\mathcal{L}(g_1, ..., g_n)$ under scale transformations $x \to x + dt\, x$ can be seen as a transformation of the coupling constants $\vec{g} \to \vec{g}'$. Scale invariance in this context means that under a scale transformation the point $\vec{g}$ remains unchanged. Scale transformations induce a sort of *motion* in the n-dimensional space of couplings. Such a space is called usually the Space of Actions. A trajectory in this space is a function $\vec{g}(t)$ (running coupling constants). The variation of the Lagrangian is $\mathcal{L}(t+dt) = \mathcal{L}(t) + \partial_\mu J^\mu dt$ where $J^\mu = x_\nu T^{\mu\nu}$ is the Nöther current associated to the scale transformation. The object $\partial_\mu J^\mu = T^\mu_\mu \equiv \Theta$ is the trace of the stress energy tensor (we shall always consider in the following a symmetrized improved stress energy tensor [7]) and scale invariance, where J^μ satisfies a continuity equation $\partial_\mu J^\mu = 0$, implies $\Theta = 0$. For a general trajectory $\vec{g}(t)$ the stress energy tensor is not traceless. The *Callan - Symanzyk beta function* is defined as

$$\beta_i(\vec{g}) \equiv \frac{dg_i}{dt} \tag{1}$$

and can be interpreted as a sort of *velocity field* of the theory in the action space. Let us consider one coupling for simplicity. Imagine to be at a certain point in the space of actions where the action of the theory is characterized by a parameter g_0. Consider this as a sort of initial condition, and study the behaviour of the theory as t increases.

- if $\beta(g_0) > 0$ then $g(t)$ is increasing in a region near g_0;
- if $\beta(g_0) < 0$ then $g(t)$ is decreasing in a region near g_0.

Points g^* where $\beta(g^*) = 0$ are called *fixed points*. They divide the space of actions (one-dimensional in this example) in distinct regions of the g parameter. The running coupling constant keeps confined in one of these regions. Independently of the exact value of g_0, in a given region, the asymptotic value of $g(t)$ for $t \to \pm\infty$ is always given by the closest values of g where $\beta(g) = 0$. There are two types of fixed points:

- a fixed point g^* is *infrared* (IR) if, starting from a value g_0 of the coupling near it, one reaches it for $t \to +\infty$. ($\beta'(g^*) < 0$).
- a fixed point g^* is *ultraviolet* (UV) if, starting from a value g_0 of the coupling near it, one reaches it for $t \to -\infty$. ($\beta'(g^*) > 0$).

In the case with more parameters all the beta functions have to annullate simultaneously in a fixed point. Given the initial conditions $\vec{g} = \vec{g}_0$ there exists a unique trajectory, as the equation (1) is a first order linear differential equation. The trajectory $\vec{g}(t)$ is often referred in the literature as the *Renormalization Group (RG) flow.*

The knowledge of the beta function is enough to reconstruct the behaviour of the theory in a surrounding of the point g_0 i.e. to have a complete description of the tangent space to the space of actions in the neighborhood of g_0. In such tangent space, the variation of the Lagrangian can be expressed as a linear combination of a base of fields identified as the derivatives of the Lagrangian with respect to the coupling constants g_i

$$\Phi_i = \frac{\partial \mathcal{L}}{\partial g_i}$$

The trace of the stress energy tensor is a field living in such tangent space

$$\Theta(x) = \frac{d\mathcal{L}}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial g_i}\frac{dg_i}{dt} = \sum_i \beta_i(\vec{g})\Phi_i(\vec{g})$$

as it is a linear combination of the fields Φ_i with coefficients given by the components of the beta function vector. It is then obvious that $\Theta = 0$ when and only when $\beta_i(\vec{g}) = 0$, which means that the fixed points of a QFT are scale invariant.

We go now to get an equation describing the variation of the N point correlation functions of generic fields $A_i(x)$ of the theory $\langle X \rangle \equiv \langle A_1(x_1), ..., A_N(x_N) \rangle$ along the RG flow. It is known in the literature as the Callan-Symanzyk equation. We present here a derivation inspired by A. Zamolodchikov [8]. The variation of correlation functions under scale transformation can be described in two equivalent ways

1. apply the variation to all fields in the correlator

$$\begin{aligned} \delta\langle X \rangle &\equiv \delta\langle A_1(x_1), ..., A_N(x_N) \rangle = \delta \int \mathcal{D}\varphi \, A_1 ... A_N \, e^{-S[\phi]} \\ &= \delta \int \mathcal{D}\varphi \, \left[\sum_k A_1 ... \delta A_k ... A_N \, e^{-S} - A_1 ... A_N \, e^{-S} \delta S \right] \end{aligned}$$

   The variation of a generic field A_k under scale transformation is given by the global dilation of coordinates and by the intrinsic variation of the field due to its ingeneering (classical) dimension D_k

$$\delta A_k = dt(x_k^\mu \partial_\mu^k + D_k)A_k$$

   The variation of the action is actually given by the insertion of the trace of the stress energy tensor into the correlator

$$\delta S = \int d^2y \Theta(y)$$

   Therefore

$$\delta\langle X \rangle = dt \left(\sum_k (x_k^\mu \partial_k^\mu + D_k)\langle X \rangle - \int d^2y \langle X\Theta(y) \rangle \right)$$

2. alternatively one can vary the fields and the action with respect to the couplings g_i

$$\delta\langle X\rangle = \sum_i \beta_i(\vec{g})\frac{\partial}{\partial g_i}\langle X\rangle$$

but

$$\frac{\partial}{\partial g_i}\langle X\rangle = \sum_k \langle A_1, ..., \frac{\partial A_k}{\partial g_i}, ..., A_N\rangle - \int \langle X\Phi_i(x)\rangle d^2x$$

Be careful, in these formulae, not to confuse summations over i, that run in the tangent to the space of actions and summations over k that run over the points x_k in the correlator. Putting together the pieces, one gets

$$\delta\langle X\rangle = \sum_i \sum_k \langle A_1, ..., \frac{\partial A_k}{\partial g_i}, ...A_N\rangle - \int d^2y\langle X\Theta(y)\rangle$$

Combining the two methods one arrives at the so called Callan - Symanzyk equation for the 1+1 dimensional RG of QFT

$$\left[\sum_k \left(x_k^\mu \frac{\partial}{\partial x_k^\mu} + \Gamma_k(\vec{g})\right) - \sum_i \beta_i(\vec{g})\frac{\partial}{\partial g_i}\right]\langle X\rangle = 0$$

where $\Gamma_k A_k(x_k) = \left(D_k + \sum_i \beta_i(\vec{g})\frac{\partial}{\partial g_i}\right) A_k(x_k)$ is the so called *anomalous dimension* of the field A_k.

In particular the dimensions of the perturbing fields Φ_i are given by

$$\begin{aligned}\Gamma(\vec{g})\Phi_i(x) &= \sum_j \gamma_i^j(\vec{g})\Phi_j(x) = \left(D_k + \tfrac{d}{dt}\right)\tfrac{\partial L}{\partial g_i} \\ &= \tfrac{\partial}{\partial g_i}(2L + \Theta) = \sum_j \left(2\delta_i^j - \tfrac{\partial \beta_j}{\partial g_i}\right)\Phi_j(x)\end{aligned}$$

Applying this to the trace of the stress energy tensor $\Theta = \sum_i \beta_i\Phi_i$ one gets $\Gamma\Theta = 2\Theta$. This, combined with Lorentz covariance, implies that also the other components of the stress energy tensor have the same anomalous dimension 2. This is also the ingeneering dimension of the stress energy tensor, which means that the stress energy tensor does not develop an anomalous dimension and keeps its classical dimension also at the quantum level. This is a fact common to all conserved currents.

The most exciting result known in the study of two-dimensional Renormalization Group is undoubtedly the Zamolodchikov *c-theorem* [9]. Here we do not reproduce the entire argument leading to this theorem (see the original paper [9] for a demonstration). We only offer the statement of the theorem in its full generality.

c-theorem: In any *unitary* QFT_2 a function $c(\vec{g})$ exists, such that:

1. it is decreasing along the RG flow: $\frac{dc}{dt} \leq 0$;

2. it is stationary in the fixed points $\vec{g} = \vec{g}^*$, i.e. $\frac{dc}{dt}\big|_{\vec{g}^*} = 0$ if and only if $\vec{g}^*$ is a fixed point of the RG flow, i.e. $\beta_i(\vec{g}^*) = 0$.

3. in such fixed points $\vec{g}^*$, the function $c(\vec{g})$ fixes completely the two-point correlators of the stress energy tensor. Introducing the coordinates

$$z = x_1 + ix_2 \qquad \bar{z} = x_1 - ix_2$$

and the notation $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$, $\Theta = T_{z\bar{z}} = T_{\bar{z}z} = T_{\mu\mu}$ one has $\Theta = 0$ because the fixed point is scale invariant and

$$\langle T(z,\bar{z})T(0,0)\rangle = \frac{c/2}{z^4} \qquad \langle \bar{T}(z,\bar{z})\bar{T}(0,0)\rangle = \frac{c/2}{\bar{z}^4} \qquad \langle T\bar{T}\rangle = 0 \tag{2}$$

where $c = \lim_{\vec{g}\to\vec{g}^*} c(\vec{g})$. The constant c is often called *central charge* and is a characteristic of the theory at the fixed point.

As we have seen that theories are RG flows from UV to IR fixed points when t goes from $-\infty$ to $+\infty$. This theorem means that, for unitary theories

$$c_{UV} \geq c_{IR}$$

which is the "simplified" form frequently mentioned in the literature.

3 Conformal Field Theory

In the previous section we have seen that the RG fixed points correspond to values of the parameters of a theory where scale invariance is guaranteed. Scale invariance implies under quite general assumptions a larger invariance, known as conformal invariance. In this section we summarize the main results in the treatment of conformal invariant theories (CFT) in two-dimensions, as developed in the fundamental work [10].

3.1 The Virasoro algebra and the Hilbert space

In two dimensions any analytic transformation of co-ordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ is a conformal transformation. The improved stress energy tensor obeys the continuity equation $\partial_\mu T^{\mu\nu} = 0$ and is symmetric $T^{\mu\nu} = T^{\nu\mu}$. If we add the requirement of scale invariance, i.e. of null trace $T^{\mu\mu} = \Theta = 0$, we end up, in two dimensions, to have only two independent components $T = T_{zz} = \frac{1}{4}(T^{11} - T^{22} + 2iT^{12})$ and $\bar{T} = T_{\bar{z}\bar{z}} = \frac{1}{4}(T^{11} - T^{22} + 2iT^{12})$, for which the continuity equations read

$$\partial_{\bar{z}} T(z,\bar{z}) = \partial_z \bar{T}(z,\bar{z}) = 0$$

expressing that T and $\bar{T}$ are functions only of z and $\bar{z}$ respectively: $T = T(z)$ and $\bar{T} = \bar{T}(\bar{z})$. The theory separates into two "non-interacting" parts with dependence on z and $\bar{z}$

respectively. In what follows we shall concentrate on the z part, although all statements can be repeated for the $\bar{z}$ part too.

The stress-energy tensor is the generator of conformal transformations, in the sense that if we take its Laurent expansion

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}} \qquad \text{i.e.} \qquad L_n = \oint_0 \frac{dz}{2\pi i} T(z) z^{n+1} \tag{3}$$

then the mode L_n generates the infinitesimal conformal transformation $z \to z + \varepsilon z^{n+1}$ (analogous expansion holds for $\bar{T}(\bar{z})$ with modes $\bar{L}_n$). In particular L_{-1} generates the translations in the z direction, $L_0 + \bar{L}_0$ generates the dilations, $L_0 - \bar{L}_0$ the rotations. One can write down conformal Ward identities, that, once integrated, give the operator product expansion (OPE) for any field with the stress-energy tensor. In particular the OPE of T with itself reads as

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular terms} \tag{4}$$

The number c is called *conformal anomaly* or often *central charge* and plays a central role in the following. Eq.(4) implies for the modes (3) the Virasoro ($\mathcal{V}$) algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{5}$$

where the constant c appears in the central extension term. The algebra of the full conformal group (do not forget the $\bar{z}$ part) is then $\mathcal{V} \otimes \bar{\mathcal{V}}$. We shall use the so called *radial quantization*, where the time is taken as $\log|z|$. In this description $L_0 + \bar{L}_0$ is the Hamiltonian, and its eigenvalues must be bounded by below. All the states of a CFT must lie in some irreducible representation of the algebra $\mathcal{V} \otimes \bar{\mathcal{V}}$. The representations we need must have L_0 and $\bar{L}_0$ eigenvalues bounded by below, i.e. they must contain a so called *highest weight state* (HWS) $|\Delta\rangle$ for which

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \qquad , \qquad L_n|\Delta\rangle = 0 \qquad , \qquad n > 0 \tag{6}$$

These representations are known as *highest weight representations* (HWR). The irreducible representations of $\mathcal{V}$ are labelled by two numbers; for HWRs these are c and Δ. We shall denote the HWRs of $\mathcal{V}$ by $\mathcal{V}_c(\Delta)$. For a given theory, c is fixed by eq.(4), then the Hilbert space $\mathcal{H}$ of the theory is built up of all possible representations $\mathcal{V}_c(\Delta)$ at fixed c, each one with a certain multiplicity:

$$\mathcal{H} = \bigoplus_{\Delta,\bar{\Delta}} \mathcal{N}_{\Delta,\bar{\Delta}} \mathcal{V}_c(\Delta) \otimes \bar{\mathcal{V}}_c(\bar{\Delta}) \tag{7}$$

If a certain $\mathcal{V}_c(\Delta) \otimes \bar{\mathcal{V}}_c(\bar{\Delta})$ does not appear, then simply $\mathcal{N}_{\Delta,\bar{\Delta}} = 0$. The numbers $\mathcal{N}_{\Delta,\bar{\Delta}}$ count the multiplicity of each representation in $\mathcal{H}$, this implies they must always be non

negative integers. They are not fixed by conformal invariance. Constrains on them arise from other physical requirements such as locality [12] or modular invariance [2].

From eq.(4) one easily computes the two point correlator of the stress energy tensor, and comparing with eq.(2) it turns out that c is exactly the value of the $c(\vec{g})$ function of the c-theorem at this particular fixed point.

Any state $|s\rangle \equiv |\Delta, \bar{\Delta}\rangle$ in the theory can be put in 1 to 1 correspondence with a field through the formula $|s\rangle = A_s(0,0)|0\rangle$, where the vacuum $|0\rangle$ is projective (i.e. $L_0, \bar{L}_0, L_{\pm 1}, \bar{L}_{\pm 1}$) invariant. In particular the HWS (6) correspond to some fields $\phi_{\Delta,\bar{\Delta}}(z,\bar{z})$ that transform under the conformal group as

$$\phi_{\Delta,\bar{\Delta}}(z,\bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{\Delta}} \phi_{\Delta}(z',\bar{z}') \tag{8}$$

They are called *primary fields*. Their OPE with the stress-energy tensor is given by

$$T(z)\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) = \frac{\Delta\phi_{\Delta,\bar{\Delta}}(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \phi_{\Delta,\bar{\Delta}}(w,\bar{w})}{z-w} + \text{regular terms} \tag{9}$$

Applying this formula to the vacuum and using (3) we go back to (6). A basis for all the other states of the theory (called *secondaries*) can be obtained by applying strings of $L_n, n<0$ to $|\Delta\rangle$. The commutation relations imply

$$L_0 L_n^k |\Delta\rangle = (\Delta + nk) L_n^k |\Delta\rangle \tag{10}$$

Therefore L_0 eigenvalues organize the space $\mathcal{V}_c(\Delta)$ (often called a *module*) so that the states lie on a "stair" whose N-th step (called the N-th *level*) has $L_0 = \Delta + N$

states	level	L_0
.....	...	...
$L_{-3}\|\Delta\rangle$, $L_{-2}L_{-1}\|\Delta\rangle$, $L_{-1}^3\|\Delta\rangle$	3	$\Delta+3$
$L_{-2}\|\Delta\rangle$, $L_{-1}^2\|\Delta\rangle$	2	$\Delta+2$
$L_{-1}\|\Delta\rangle$	1	$\Delta+1$
$\|\Delta\rangle$	0	Δ

(11)

All the fields corresponding to the HWR $\mathcal{V}_c(\Delta)$ are said to be in the *conformal family* $[\phi_\Delta]$ generated by the primary field ϕ_Δ.

3.2 Unitarity

We said that for a given theory, c is fixed by eq.(4). Conversely, we can ask how many theories are there at a certain value of c. The answer to this question for all c's is what is called *classification* of CFTs.

As a first step towards this classification, one can ask to describe how many HWR $\mathcal{V}_c(\Delta)$ are there at a fixed c. In general any value of c and Δ could work, but if we require *unitarity*, i.e. absence of negative norm states, then we can state the following results[11]:

1. for $c < 0$ no representation is unitary

2. for $0 \leq c < 1$ the following set of $\mathcal{V}_c(\Delta)$ is unitary

$$c = 1 - \frac{6}{p(p+1)} \qquad , \qquad p = 2, 3, 4, ... \tag{12}$$

$$\Delta = \Delta_{rs} = \frac{[(p+1)r - ps]^2 - 1}{4p(p+1)} \qquad 1 \leq s \leq r \leq p-1 \qquad r, s \in \mathbb{Z} \tag{13}$$

3. for $c \geq 1$ all representations are unitary

Therefore a theory containing negative Δ's or c is automatically non-unitary. Unitarity is an essential requirement in string theory. Also many statistical systems enjoy it, but there are well known cases (percolation, Lee-Yang edge singularity) where unitarity does not hold (i.e. the Hamiltonian is not real).

3.3 The OPE-algebra and correlation functions

All fields in a CFT are expected to obey a closed algebra under OPE

$$A_i(z, \bar{z}) A_j(0, 0) = \sum_k B_{ij}^k(z, \bar{z}) A_k(0, 0) \tag{14}$$

where i, j, k run over all fields (primaries and secondaries) of the theory. Conformal invariance (via eq.(9)) reduces the problem to the knowledge of the OPE-algebra among primary fields and puts constraints on the form of the functions $B_{ij}^k(z, \bar{z})$. The OPE-algebra of primary fields reads as

$$\phi_i(z, \bar{z}) \phi_j(0, 0) = \sum_k C_{ij}^k z^{\Delta_k - \Delta_i - \Delta_j} \bar{z}^{\bar{\Delta}_k - \bar{\Delta}_i - \bar{\Delta}_j} [\phi_k(0, 0)] \tag{15}$$

where now the indices i, j, k run over all primaries of the theory and $[\phi_k(0, 0)]$ means contribution from the whole conformal family $[\phi_k]$, which can be seen as an expansion over all secondaries of $[\phi_k]$, whose coefficients are also (in principle) fixed by conformal invariance. The only objects that remain unfixed are the *structure constants* C_{ij}^k. Were these known, one could reduce via iterative applications of OPEs (15) all the correlators among primaries to 2 and 3 point functions, which are fixed by projective invariance

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle = \delta_{12} z_{12}^{2\Delta_1} \bar{z}_{12}^{2\bar{\Delta}_i} \tag{16}$$

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \phi_3(z_3, \bar{z}_3) \rangle = C_{12}^3 z_{12}^{\gamma_{12}} z_{13}^{\gamma_{13}} z_{23}^{\gamma_{23}} \bar{z}_{12}^{\bar{\gamma}_{12}} \bar{z}_{13}^{\bar{\gamma}_{13}} \bar{z}_{23}^{\bar{\gamma}_{23}} \tag{17}$$

where $z_{ab} = z_a - z_b$, $\gamma_{ab} = \Delta_c - \Delta_a - \Delta_b$ with $a \neq b \neq c$ and $a, b, c = 1, 2, 3$. Correlators among secondaries could be reduced to correlators among primaries via the OPE (9). So, at least in principle, all correlators in the theory are computable: the theory is exactly solvable. Unfortunately in most cases we do not know the constants C_{ij}^k. Constraints on C_{ij}^k come from the requirement of associativity of the OPE-algebra, which is equivalent to ask *duality* of the 4-point functions. So in general C_{ij}^k can be computed if we know the 4-point functions.

3.4 Null vectors and minimal models

It can happen that in a certain HWR a secondary state $|\chi\rangle$ (to be called *null vector*) satisfies $L_0|\chi\rangle = (\Delta + N)|\chi\rangle$ and $L_n|\chi\rangle = 0$, $n > 0$. This means that $|\chi\rangle$ behaves like a primary state. This apparent contradiction is solved by saying that the HWR is not irreducible. Taking $|\chi\rangle \equiv 0$ (that can always be done consistently as it is easy to prove that $\langle s|\chi\rangle = 0$ for all $|s\rangle \in$ HWR, including $|\chi\rangle$ itself) we get rid of the representation embedded in the HWR. If there are more than one null vector, we repeat this procedure until we have "cleaned" the representation from all null vectors. What remains is a true irreducible module $\mathcal{V}_c(\Delta)$ that can be used to build up a CFT.

When null vectors appear, they give constraints in the form of partial differential equations on the correlators of primary fields. Indeed, consider a correlator $\langle\chi(z)\phi_1(z_1)...\phi_n(z_n)\rangle$ where $\phi_i(z_i)$ are primary fields and $\chi(z)$ is a null vector in the conformal family $[\phi]$. Then $\chi(z)$ is a secondary of $\phi(z)$ and eq.(9) (or more precisely the conformal Ward identity obtained by it) implies the existence of a suitable differential operator $\mathcal{D}$ such that $\langle\chi(z)\phi_1(z_1)...\phi_n(z_n)\rangle = \mathcal{D}\langle\phi(z)\phi_1(z_1)...\phi_n(z_n)\rangle$. On the other hand $\chi \equiv 0$, so that we have

$$\mathcal{D}\langle\phi(z)\phi_1(z_1)...\phi_n(z_n)\rangle = 0 \tag{18}$$

which is a differential equation to be satisfied by the correlators among primary fields. This constraint can be so powerful to select a finite number of primary fields under which the OPE-algebra closes. In this case only a finite set of $\mathcal{V}_c(\Delta)$ are used to build up the model. We shall speak in this case of a *minimal* model of $\mathcal{V}$. Minimal models exist for $c < 1$ only (there is a theorem by Cardy [2] that prevents from the possibility to construct a model with finite number of $\mathcal{V}_c(\Delta)$ for $c \geq 1$). More precisely unitary minimal models can exist only for the values of $c = \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{6}{7}, ...$ given by the formula (12) and they can be built up using only the $\mathcal{V}_c(\Delta)$ representations such that Δ is contained in the *Kac-table* given by eq.(13).

The differential equations (18) reduce, in the case of 4-point functions and after mapping $z_1 \to 0, z_2 \to 1, z_4 \to \infty$, to ordinary fuchsian differential equations in $z \equiv z_3$. In the case of minimal models a solution to these equations can be given [12]. This means that the C_{ij}^k can be computed, and so exact solvability of the minimal models is ensured.

3.5 CFT on a cylindrical geometry and FSE

The mapping of the complex plane into a strip of thickness L with periodic boundary conditions is a conformal transformation, realized by the mapping

$$z = e^{\frac{2\pi}{L}u} \quad , \quad \bar{z} = e^{\frac{2\pi}{L}\bar{u}} \tag{19}$$

with $u = \tau + i\sigma$ and $\bar{u} = \tau - i\sigma$, where $\tau \in]-\infty, +\infty[$ is the cylinder "time" and $\sigma \in [0, L[$ the cylinder "space". Under conformal mappings, the stress energy tensor transforms as

$$T(z) = \left(\frac{\partial u}{\partial z}\right)^2 T(u) + \frac{c}{12}\left[\frac{\partial_z^3 u}{\partial_z u} - \frac{3}{2}\left(\frac{\partial_z^2 u}{\partial_z u}\right)^2\right]$$

Consequently, the modes of T on the plane map on those on the cylinder according to

$$T(u) = -\frac{c}{24} + \sum_{n\in\mathbb{Z}} L_n e^{inu}$$

i.e. they become Fourier modes on the cylinder. The radial quantization in z briefly mentioned above becomes in this cylindrical context a usual time ordering quantization. Periodic boundary conditions mean $T(u+L) = T(u)$. The physical interpretation of modes on the cylinder is that total energy and momentum of the system are given by the zero modes

$$E = \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12}) \quad , \quad P = \frac{2\pi}{L}(L_0 - \bar{L}_0)$$

Therefore a state $|\Delta, \bar{\Delta}\rangle$ on the cylinder represents a state of given energy and momentum

$$E = \frac{2\pi}{L}(\Delta + \bar{\Delta} - \frac{c}{12}) \quad , \quad P = \frac{2\pi}{L}(\Delta - \bar{\Delta})$$

In particular the vacuum of a unitary theory ($\Delta = \bar{\Delta} = 0$) has energy

$$E_{vac} = -\frac{\pi c}{6L}$$

given by the Casimir effect of being on a finite and periodic geometry. We see that the constant c has the physical meaning of measuring this Casimir effect.

3.6 Integrability of CFT

CFT is an integrable QFT. This can be shown by explicitly constructing an infinite set of conserved charges. Define the charges

$$I_{s+1} = \frac{1}{L}\int_0^L d\sigma\, T_s(u) = \oint dz\, T_s(z)$$

where the currents $T_s(u)$ are time ordered (in τ time) polynomials in T and its derivatives at level s in the Verma module of the identity operator. One can show that if s is odd, all the T_s fields at level s can be arranged to be total derivatives of some T_{s-1} and therefore their integrals I_{s+1} are trivially zero. Nontrivial integrals of motion generated by T exist only for even s. At a given level there can be more than one monomial in T and its derivatives, the most general current will be a linear combination of these. If we require that the corresponding charges commute with all the ones at smaller values of s then the coefficients turn out to be fixed in a unique way, thus giving a sequence of currents, one for each even level s, that are in involution, i.e mutually commute. This actually defines an integrable QFT. Here are the first few currents of this kind.

$$\begin{gathered} T_2(u) = T(u) \\ T_4(u) =: T(u)^2 : \\ T_6(u) =: T(u)^3 : + \tfrac{c+2}{12} : (\partial_u T(u))^2 : \end{gathered}$$

etc... Of course a similar sequence of integrals $\bar{I}_{s+1}$ holds also for $\bar{T}$. Note that the first current is T itself. Therefore, energy and momentum are conserved quantities belonging to this set, and build up a vector of spin 1. All the other charges, as they commute with the Hamiltonian, are also conserved charges in an infinite sequence of higher and higher odd spins.

Of course, if the model possesses higher symmetries (like in W-algebras or in WZW models) there can be other sets of conserved currents and the sequence of spin of the conserved charges could be different.

3.7 c=1 Conformal Field Theory

A particular example of CFT that will be used frequently in the sequel of this review is the case of the so called *massless free boson* compactified on a target space of a circle of radius R. Here we give a brief summary of such a $c = 1$ CFT. The Lagrangian is taken to be

$$\mathcal{L} = \frac{1}{8\pi} \int_0^L \partial_\mu \varphi(\sigma,\, \tau) \partial^\mu \varphi(\sigma,\, \tau) d\sigma \;,\; \sigma \in [0, L] \;, \tag{20}$$

where L is the spatial volume (i.e. the theory is defined on a cylindrical space-time with circumference L). The superselection sectors are classified by the $\widehat{U(1)}_L \times \widehat{U(1)}_R$ Kac-Moody symmetry algebra, generated by the currents

$$J(z) = i\partial_z \varphi \,, \quad \bar{J}(\bar{z}) = i\partial_{\bar{z}} \varphi \,.$$

where we have mapped the cylinder back to the plane according to eq.(19). The left/right moving energy-momentum tensor is given by

$$T(z) = \frac{1}{8\pi} \partial_z \varphi \partial_z \varphi = \sum_{k=-\infty}^{\infty} L_k z^{-k-2} \quad , \quad \bar{T}(\bar{z}) = \frac{1}{8\pi} \partial_{\bar{z}} \varphi \partial_{\bar{z}} \varphi = \sum_{k=-\infty}^{\infty} \bar{L}_k \bar{z}^{-k-2}$$

The coefficients L_n and $\bar{L}_n$ of the Laurent expansion of these fields generate two mutually commuting Virasoro algebras. If the quasi-periodic boundary conditions are required for the boson

$$\varphi(\sigma + L, \tau) = \varphi(\sigma, \tau) + 2\pi m R \,, \quad m \in \mathbb{Z} \,,$$

then the sectors are labelled by a pair of numbers $(n,\, m)$, where $\frac{n}{R}$ (n is half integer because of the locality, see later) is the eigenvalue of the total field momentum π_0

$$\pi_0 = \int_0^L \pi(\sigma, \tau) d\sigma \,, \quad \pi(\sigma, \tau) = \frac{1}{4\pi} \partial_\tau \varphi(\sigma, \tau) \,,$$

and m is the winding number, i.e. the eigenvalue of the topological charge Q defined by

$$Q = \frac{1}{2\pi R}\int_0^L \partial_\sigma \varphi(\sigma,\tau) d\sigma\,.$$

In the sector with quantum numbers $(n,\, m)$, the scalar field is expanded in modes as follows:

$$\begin{array}{rl} \varphi(\sigma,\tau) = & \phi(z) + \bar\phi(\bar z)\,, \\ \phi(z) = & \frac{1}{2}\varphi_0 - ip_+ \log z + i\sum_{k\neq 0}\frac{1}{k}a_k z^{-k}\,, \\ \bar\phi(\bar z) = & \frac{1}{2}\varphi_0 - ip_- \log \bar z + i\sum_{k\neq 0}\frac{1}{k}\bar a_k \bar z^{-k}\,, \end{array}$$

where the left and right moving field momenta $p_\pm$ (which are in fact the two $U(1)$ Kac-Moody charges) are given by

$$p_\pm = \frac{n}{R} \pm \frac{1}{2} mR\,. \tag{21}$$

The Virasoro generators take the form

$$L_n = \frac{1}{2}\sum_{k=-\infty}^{\infty} :a_{n-k}a_k:\,, \quad \bar L_n = \frac{1}{2}\sum_{k=-\infty}^{\infty} :\bar a_{n-k}\bar a_k:\,,$$

where the colons denote the usual normal ordering, according to which the oscillator with the larger index is put to the right.

The ground states of the different sectors $(n,\, m)$ are created from the vacuum by the (Kac-Moody) primary fields, which are vertex operators of the form

$$V_{(n,\,m)}(z,\overline{z}) =: \exp i(p_+\phi(z) + p_-\bar\phi(\overline{z})) :\ . \tag{22}$$

The left and right conformal weights of the field $V_{(n,\,m)}$ (i.e. the eigenvalues of L_0 and $\bar L_0$) are given by the formulae

$$\Delta^\pm = \frac{p_\pm^2}{2}. \tag{23}$$

The Hilbert space of the theory is given by the direct sum of the Fock modules built over the states

$$|n,\, m\rangle = V_{(n,\,m)}(0,0)\,|vac\rangle\ , \tag{24}$$

with the help of the creation operators a_{-k} , $\bar{a}_{-k}$ $k > 0$:

$$\mathcal{H} = \bigoplus_{(n,\,m)} \{a_{-k_1} \ldots a_{-k_p} \bar{a}_{-l_1} \ldots \bar{a}_{-l_q} |n,\, m\rangle,\, k_1, \ldots\, k_p,\, l_1, \ldots\, l_q \in \mathbb{Z}_+\}$$

The boson Hamiltonian on the cylinder is expressed in terms of the Virasoro operators as

$$H_{CFT} = \frac{2\pi}{L}\left(L_0 + \overline{L}_0 - \frac{c}{12}\right) , \tag{25}$$

where the central charge is $c = 1$. The generator of spatial translations is given by

$$P = \frac{2\pi}{L}\left(L_0 - \bar{L}_0\right) . \tag{26}$$

The operator $L_0 - \bar{L}_0$ is the conformal spin which has eigenvalue nm on the primary field $V_{(n,\,m)}$.

One can also introduce twisted sectors using the operator $\mathcal{T}$ that performs spatial translations by L: $x \to x + L$. The primary fields $V_{(n,\,m)}$ as defined above satisfy the periodicity condition $\mathcal{T} V_{(n,\,m)} = V_{(n,\,m)}$. If the more general twisted boundary condition labelled by a real parameter ν is required

$$\mathcal{T} V_{(n,\,m)} = \exp\left(i\nu Q\right) V_{(n,\,m)} ,$$

then it is possible to generate superselection sectors for which $n \in \mathbb{Z} + \frac{\nu}{2\pi}$.

It is important to stress that a particular $c = 1$ CFT is specified by giving the spectrum of the quantum numbers (n, m) (and the compactification radius R) such that the corresponding set of vertex operators (and their descendants) forms a *closed and local* operator algebra. The locality requirement is equivalent to the fact that the operator product expansions of any two such local operators is single valued in the complex plane of z. This condition, which is weaker than the modular invariance of the CFT, is the adequate one since the theory is considered on a space-time cylinder and we do not wish to define it on higher genus surfaces.

By this requirement of locality, it was proved in [13] that there are only two maximal local subalgebras of vertex operators: $\mathcal{A}_b$ generated by the vertex operators

$$\{V_{(n,\,m)} : n,\, m \in \mathbb{Z}\} ,$$

and $\mathcal{A}_f$ generated by

$$\{V_{(n,\,m)} : n \in \mathbb{Z},\, m \in 2\mathbb{Z}\ or\ n \in \mathbb{Z} + \frac{1}{2},\, m \in 2\mathbb{Z} + 1\} .$$

Other sets of vertex operators can be built, but the product of two of them gives a nonlocal expression.

4 Perturbed Conformal Field Theory

In the previous sections we have seen that a QFT can be thought as a RG flow in the space of actions, and its UV and IR points correspond to scale invariant CFT. We have then concentrated on CFT and explored its properties. Armed with this new insight we may wonder if the CFT data are able to predict some piece of information about the theory out of the fixed points.

We may think to define a QFT as a deformation of a critical CFT by some operators [14], i.e. to assume as action the following expression

$$S = S_{CFT} + \sum_{i=1}^{n} \lambda_i \int d^2x \Phi_i(x) \tag{27}$$

Of course, the class of QFT_2 is larger than the one described by this sort of action. Nevertheless, this class of the so called *Perturbed Conformal Field Theories* (PCFT) is very important because most of the predictions can be done working in this realm and many applications to the vicinity of critical points in the theory of critical phenomena can be described by this class of actions.

Lorentz invariance in 1+1 dimensions is equivalent to rotational invariance in the 2 dimensional space once a Wick rotation is performed. Hence, to achieve Lorentz invariance we must require that the fields Φ_i are scalars under 2 dimensional rotations, i.e. their conformal spin $\Delta_i - \bar{\Delta}_i$ must be null, which implies $\Delta_i = \bar{\Delta}_i$. Their anomalous dimensions are therefore $2\Delta_i$.

4.1 Relevant, Irrelevant and Marginal Perturbations

Let us consider the simple case of one single perturbing field Φ, of left conformal dimension Δ. The coupling λ is therefore dimensionful, with scaling dimension $y = 2 - 2\Delta$. Renormalization is needed [15], and can be achieved by introducing a dimensionless coupling g, a mass scale μ and a renormalized field $\Phi(x, g)$ by the relations

$$g = \mu^{-y} Z_g \lambda \quad , \quad \Phi(x) = \sqrt{Z_\Phi}\Phi(x, g)$$

where Z_g and Z_Φ are renormalization prefactors collecting all infinite contributions.

$g = 0$ is a fixed point, because the interaction perturbing term disappears and one is left with the pure S_{CFT}. The field $\Phi = \frac{\partial L}{\partial g}$ must have a limit in the unperturbed CFT, say Φ_0, with

$$\Gamma(0)\Phi_0 = \gamma(0)\Phi_0 = 2\Delta\Phi_0 \tag{28}$$

We are asking which are the conditions on the field Φ that guarantee that the fixed point at $g = 0$ is an UV one, from which the RG flow actually gets out and, increasing t goes to some IR destiny. To have this, g must *increase* towards an IR fixed point g^*, and the

anomalous dimension of Φ changes according to the already seen formula

$$\Gamma(g)\Phi = \left(2 - \frac{\partial\beta}{\partial g}\right)\Phi \tag{29}$$

Putting together eqs.(28) and (29) we can compute the first order of the beta function: $\beta(g) = yg + O(g^2)$. The nature of the fixed point (UV or IR) is chosen by the sign of the first derivative of $\beta(g)$, as we saw in section 2. $y = \beta'(0) > 0$ only if $\Delta < 1$.

Fields are classified with respect to the RG group as

- *relevant* if $\Delta < 1$. If such a field perturbs a conformal action, it creates exactly the situation described above, i.e. the theory starts to flow along a RG trajectory going to some IR destiny.
- *irrelevant* if $\Delta > 1$. Such fields correspond to non renormalizable perturbations which describe the neighborhood of IR non trivial fixed points. Usually one speaks of an *attraction* field better than a perturbing one. We shall not deal with this case in the following, but the interested reader may consult, for example [16] to see some possible applications of this situation.
- *marginal* if $\Delta = 1$. These are further distinguished by the behaviour of higher derivatives of the beta function
  - if they stay marginal at all orders in perturbation theory, they are called *truly marginal*, and describe deformation of a CFT leading to another CFT with the same value of the central charge. They identify a submanifold of CFT in the space of actions.
  - if instead the anomalous dimension becomes smaller than 2 at higher orders in perturbation theory, the fields are called *marginally relevant*. They provide a perturbation of CFT defining a non scale invariant theory, but the perturbation theory is much more complicated and in particular the smoothness of the deformation of the Hilbert Space is no more valid. This sort of UV limit, that Lüscher defined as *singular* many years ago, is typical of the asymptotically free theories and hence very interesting for the properties that such theories share with QCD.
  - finally there can be some marginally irrelevant fields, defining a non renormalizable attraction to an IR no trivial point of particular theories

4.2 Conformal Perturbation Theory

In a PCFT it is natural to define a Conformal Perturbation Theory (CPT). This is a perturbation theory where the unperturbed part of the action is not taken, as usual, to

be the free part, but better the CFT at UV. A general correlator $\langle X \rangle$ of a string of fields $X = A_1 ... A_N$ expresses as

$$\langle X \rangle = \int \mathcal{D}\varphi \, X e^{-S[\varphi]} = \int \mathcal{D}\varphi \, X e^{-S_{CFT} - \lambda \int d^2 x \Phi(x)}$$

and expanding in powers of λ one can express $\langle X \rangle$ as a series of conformal correlators (in principle computable by CFT techniques) with the insertion of the perturbing field

$$\langle X \rangle = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int \langle X \Phi(x_1) ... \Phi(x_k) \rangle_{CFT} d^2 x_1 ... d^2 x_k$$

The possible IR divergences of such an expansion are cured if we put the theory on a finite geometry, e.g. on a periodic cylinder as above. A full treatment and detailed expressions for the coefficients of the CPT on the cylinder are given in [17]. CPT turns out to be UV finite if $y > 1$ i.e. $\Delta < \frac{1}{2}$. For $0 < y \leq 1$ only a *finite* number of terms are divergent: the action (27) is *super-renormalizable* in this case. Once the divergences cured (usually analytic continuation in y does the job cleanly) the series are convergent within a finite radius (which is not the case in general for perturbation series of higher dimensional QFT). It can even happen that the IR point g^* falls into the convergence region, like it has been shown in [8, 18] for the minimal models perturbed by $\Phi_{1,3}$ at large p. In this case, one can do predictions on the RG flow and follow scaling functions of physical quantities by use of the CPT. In general, the radius of convergence is too small to contain the IR region of the RG flow. Nonperturbative effects take place and one has to resort to alternative methods of calculation to study scaling functions in the IR regime in order to relate them to the particle content of the theory.

For $y = 0$ one exactly gets the case of an asymptotically free QFT. The perturbing operator is marginally relevant and the theory is renormalizable but not super-renormalizable, i.e. there are in general infinities at each order, but they are cured by the insertion of a finite number of counterterms. Finally, for $y < 0$ the theory is non-*renormalizable* and in general there are new infinities and new counterterms to be added at any order, thus loosing predictability. However, in the case of *integrable* theories, one can go a bit further [16] in restricting the form of counterterms and getting some physical information on the attracting field (these theories better describe the attraction to an IR point of a RG flow by some irrelevant operator).

4.3 Integrability in PCFT

We have seen that any CFT is an integrable theory and possesses an infinite set of mutually commuting charges. The perturbation (27) usually destroys such conservation laws. It can happen, however, that a subset of the local charges of section 3.6 keeps conserved because the modification of the current due to the perturbation can still be arranged as total derivative, thus recovering the continuity equation even off-criticality. In more precise

words, the equation $\partial_{\bar{z}} T_s(z) = 0$ valid in the unperturbed CFT is no longer valid off-criticality. It is deformed by contributions coming from the perturbation series

$$\partial_{\bar{z}} T_s = \lambda R^{(1)}_{s-1} + \lambda^2 R^{(2)}_{s-1} + ... + \lambda^N R^{(N)}_{s-1} \tag{30}$$

The fields $R^{(n)}_{s-1}$ must have, in the super-renormalizable case, a limit in the unperturbed CFT, defined by some field in the algebra of fields of the CFT, with conformal dimensions $(\Delta^{(n)}, \bar{\Delta}^{(n)})$ appearing in the Kac table of primaries or secondaries. Dimensional balancing of eq.(30) gives

$$\Delta^{(n)} = s - n(1-\Delta) \qquad \bar{\Delta}^{(n)} = 1 - n(1-\Delta) \tag{31}$$

where Δ is the dimension of the perturbing field Φ. This implies that the series (30) actually truncates at some integer N, as conformal dimensions in a CFT are bounded by below. In general there is only one field in the Kac table compatible with (31) appearing in (30). In this case we speak of a no-resonance perturbation. Concentrating on this simple case, where only $R^{(1)}_{s-1}$ appears with conformal dimensions $(s-1+\Delta, \Delta)$, we see that such field must be a secondary of the perturbing field Φ of spin $s-1$. It is obtained by applying only left modes L_{-n} to Φ. If the combination of modes can be put in the form of

$$L_{-1} \cdot (\text{any combination of strings of } L_{-n}\text{'s})$$

then, as $L_{-1} \equiv \partial_z$, one can see that $\lambda R^{(1)}_{s-1} = \partial_z \Theta_{s-2}$ for some field Θ_{s-2} also in the secondaries of Φ but one level below. Then eq.(30) boils down to a genuine continuity equation

$$\partial_{\bar{z}} T_s = \partial_z \Theta_{s-2}$$

guaranteeing a conservation of the charge

$$I_{s+1} = \oint dz\, T_s(z,\bar{z}) + \oint d\bar{z}\, \Theta_{s-2}(z,\bar{z})$$

If such a situation is realizable or not depends on the nature of the perturbation Φ. A. Zamolodchikov [14] has given a sufficient but not necessary condition to ensure the presence of a conserved current, based on the analysis of the dimensions of subspaces of the Verma modules of Φ and the identity. For details see the original article [14].

Integrability of a two-dimensional massive QFT has strong implications on the properties of its S-matrix [32].

1. the number of particles is conserved in the scattering process, there is no particle production. Only exchanges of internal quantum numbers are allowed among particles in a multiplet;

2. the final set of momenta coincides with the initial one: $\{p_1, ..., p_n\}_{in} \equiv \{p'_1, ..., p'_n\}_{out}$;

3. the S-matrix for a process with n incoming and n outgoing particles factorizes into S-matrices of 2 incoming and 2 outgoing particles. To guarantee that the order of this factorization is irrelevant, the 2 particle S-matrices have to satisfy a factorization equation known as *Yang-Baxter* equation (here 1,2,3 label the particles and $\theta_{ij} = \theta_i - \theta_j$, θ_i being the rapidity of particle i)

$$S_{i_1 i_2}^{k_1 k_2}(\theta_{12}) S_{k_1 i_3}^{j_1 k_3}(\theta_{13}) S_{k_2 k_3}^{j_2 j_3}(\theta_{23}) = S_{k_1 k_2}^{j_1 j_2}(\theta_{12}) S_{i_1 k_3}^{k_1 j_3}(\theta_{13}) S_{i_2 i_3}^{k_2 k_3}(\theta_{23})$$

The factorizable S-matrix of an integrable PCFT can often be conjectured by the knowledge of integrals of motion, that constrain the bootstrap equations and by the Yang-Baxter Equation itself, that imposes severe constraints on the matricial form of $S_{ab}^{cd}(\theta)$. However, this conjecture has to be verified against independent checks. The particle description of the Hilbert space must be linked with the conformal description valid in the vicinity of the critical point, otherwise one cannot assert that the two theories defined as perturbation of a CFT and as factorized scattering theory are the same.

4.4 Scaling functions

One possible bridge between the two formulations – perturbed CFT versus Factorizable Scattering Theory – is given by FSE, more specifically by the use of the so called *scaling functions*. We have seen that the energy of a state $|i\rangle$ on a cylinder of circumference L in CFT is given by

$$E_i = -\frac{\pi c_i}{6L} \quad \text{with} \quad c_i = c - 12(\Delta_i + \bar{\Delta}_i) \tag{32}$$

It is obvious that when we move outside the critical point such energy level has a different dependence on L. However, the overall $1/L$ dependence is fixed by dimensional reasons, so all the variation of the dependence must be confined in a dimensionless function that we are free to normalize in such a way that in the limit $L \to 0$ it reproduces eq.(32). Being dimensionless, such a function can depend on L only in a dimensionless way, i.e. we are forced to introduce a mass parameter m to compensate for the physical dimension of L. The dependence will be on a dimensionless parameter $l = mL$. This is very reminiscent of what already illustrated in the introduction and the discussions made there about the UV and IR limits recovered for $l \to 0$ and $l \to \infty$ respectively of course apply here as well. They show that for each state $|i\rangle$ in the theory there is a function $c_i(l)$ attached to this state in such a way that its energy is

$$E_i(L) = -\frac{\pi c_i(l)}{6L}$$

such that

- $\lim_{l\to 0} c_i(l) = c_i = c - 12(\Delta_i + \bar{\Delta}_i)$ i.e. it reconstruct conformal data in the limit $l \to 0$;

- it can help, in a manner that will become apparent later, to extract scattering data in the limit $l \to \infty$

The forced introduction of a mass scale parameter should not surprise: we are actually going out of a critical CFT invariant point into a theory which is not scale invariant. It is therefore natural that a mass scale should be introduced.

To such functions $c_i(l)$ the name of *scaling functions* is usually given. The problem to bridge between the PCFT and the Factorized Scattering formulations of an integrable QFT_2 is recast into the computation of such scaling functions.

Of course, one way to compute them, also in non-integrable theories, is to resort to CPT. However, as already commented, the validity of CPT is normally confined to a small radius of convergence that does not allow to make contact with the IR region where scattering theory is apparent. This justifies the seek for the development of nonperturbative methods.

- The *Truncated Conformal Space Approach* (TCSA) [33] consists in diagonalizing the truncated ($E < E_{cut}$) Hamiltonian $H_{CFT} + V_{pert}$ numerically. It is non-perturbative, and applicable even to non-integrable QFT. However, it is affected by truncation errors that increase with l and, being totally numeric, does not allow for any analytic control of the scaling functions.

- The *Thermodynamic Bethe Ansatz* (TBA) [34] implements the thermodynamics of a gas of particles governed by the given S-matrix. The free energy can be exactly computed in terms of a set of coupled non-linear integral equations. Exchanging the role of time and space, such free energy can be reinterpreted as a FSE Casimir vacuum energy where the role of L is played by the temperature. Recently also excited states have been accessed with this method [36], that has been very popular for the last ten years. The treatment of the vast literature on this method is out of the scope of this review.

- In the next sections we shall illustrate an alternative method to obtain scaling functions for an integrable model starting directly from its definition as a QFT regularized on a lattice. This method is know in QFT as *Destri De Vega Non-Linear Integral Equation* (NLIE) from light cone Bethe Ansatz [26], but similar equations have also been derived in Condensed Matter Theory [35].

4.5 Sine-Gordon / massive Thirring models as PCFT

The sine-Gordon model, which has been well known at the classical level for the late fifty years, plays also an important role in quantum theory, thanks to its particular properties of non-linearity and integrability. It has been successfully applied to very different sectors of Mathematics and Physics, ranging from partial differential equation theory to particle physics or solid state physics. Recent applications of the classical model are related to nonlinear optics (resonant dielectric media) and optical fibers, magnetic properties of polymers, propagation of waves in crystals, etc... Interesting applications of the quantum

model are related to Kondo effect and to the thermodynamics of some chemical compound, as Copper-Benzoate $Cu(C_6D_5COO)_2{\cdot}3DO_2$ [19]. At the same time, the quantum theory shows a phenomenology that is similar to the Skyrme model used before QCD era to describe barions and strong interactions.

The most relevant properties of the model are

- at a classical level, all the solutions of the equations of motion are known (exact integrability via inverse scattering method)
- the classical solutions describe solitons, antisolitons and bound states (breathers); in a scattering process this solutions are transparent (it is the mathematical meaning of "soliton")
- it admits, both at a classical and at the quantum level, a countable infinite set of conserved charges
- the quantization of the theory describes an interacting particle with its antiparticle and, in a certain (attractive) regime, bound states
- the S matrix has been exactly determined; only elastic scattering processes can take place (i.e. no particle production), that is the quantum analog of the classical transparency of solitons

The minkowskian Lagrangian of sine-Gordon theory is given by

$$\mathcal{L}_{sG} = \int_{-\infty}^{+\infty} \left(\frac{1}{2} \partial_\nu \Phi \partial^\nu \Phi + \frac{\mu^2}{\beta^2} : \cos(\beta\Phi) : \right) dx \,, \tag{33}$$

where Φ denotes a real scalar field, while that of the massive Thirring theory is of the following form:

$$\mathcal{L}_{mTh} = \int_{-\infty}^{+\infty} \left(\bar{\Psi}(i\gamma_\nu \partial^\nu + m_0)\Psi - \frac{g}{2} \bar{\Psi}\gamma^\nu \Psi \bar{\Psi}\gamma_\nu \Psi \right) dx \,, \tag{34}$$

describing a current-current selfinteraction of a Dirac fermion Ψ. It is known that the two theories are deeply related provided their coupling constants satisfy

$$\frac{\beta^2}{4\pi} = \frac{1}{1 + g/\pi} \,.$$

For comparison with the Destri-de Vega nonlinear integral equation, it is important to deal with cylindrical sine-Gordon and massive Thirring, i.e. the integrals in (33, 34) must be taken in the interval $[0, L]$.

The cos term in (33) can be considered as a perturbation of the $c = 1$ free boson compactified on a cylinder, as described in section 3.7. Similarly the massless ($m_0 = 0$)

Thirring model is a $c = 1$ conformal field theory and the mass term plays the role of a perturbation. From Coleman's paper [20] it is known that correlation functions of the perturbing fields $\bar{\Psi}\Psi$ and $:\cos\beta\Phi:$ are identical, then both models can be considered as the perturbations of a $c = 1$ compactified boson by a potential V :

$$S_{sG/mTh} = S_{CFT} + V \quad , \quad V = \lambda \int \left(V_{(1,0)}(z,\overline{z}) + V_{(-1,0)}(z,\overline{z}) \right) d^2x \, , \tag{35}$$

which is related to the bosonic Lagrangian (33) by the following redefinitions of the field and the parameters:

$$\varphi = \sqrt{4\pi}\Phi \, , \quad R = \frac{\sqrt{4\pi}}{\beta} \, , \quad \lambda = \frac{\mu^2}{2\beta^2} \, . \tag{36}$$

For later convenience, a new parameter p can be defined by

$$p = \frac{\beta^2}{8\pi - \beta^2} = \frac{1}{2R^2 - 1} \, . \tag{37}$$

The point $p = 1$ (i.e. $g = 0$) is the free fermion point, corresponding to a massive Dirac free fermion. The particle spectrum of Sine-Gordon for $p > 1$ is composed by the soliton (s) and its antiparticle, the antisoliton ($\bar{s}$). It is known as repulsive regime because no bound states can take place. $p < 1$ is the attractive regime, because s and $\bar{s}$ can form bound states that are known as breathers. The values $p = \dfrac{1}{k}$, $k = 1, 2, \ldots$ are the thresholds where a new bound state appears. The potential term becomes marginal when $\beta^2 = 8\pi$ which corresponds to $p = \infty$. The perturbation conserves the topological charge Q, which can be identified with the usual topological charge of the Sine-Gordon theory and with the fermion number of the mTh model.

Mandelstam [21] showed that a fermion operator satisfying the massive Thirring equation of motion can be constructed as a nonlocal functional of a pseudoscalar field (boson) satisfying the sine-Gordon equation. But the fermion and the boson are not relatively local and then do not create the same particle (the two theories are not equivalent).

The difference between them is that they correspond to the perturbation by the same operator of the two *different local* c=1 CFTs $\mathcal{A}_b$ and $\mathcal{A}_f$ as in 3.7. The short distance behaviour of the sG theory is described by the local operator algebra $\mathcal{A}_b$, while the primary fields of the UV limit of mTh theory are $\mathcal{A}_f$.

Note that the two algebras share a common subspace with even values of the topological charge, generated by $\{V_{(n,\,m)} : n \in \mathbb{Z},\, m \in 2\mathbb{Z}\}$, where the massive theories described by the Lagrangians (33) and (34) are identical. The well known proof by Coleman about the equivalence of the two theories [20] holds exactly in this subspace. Figure 3 shows the four sectors where all the vertex operators live.

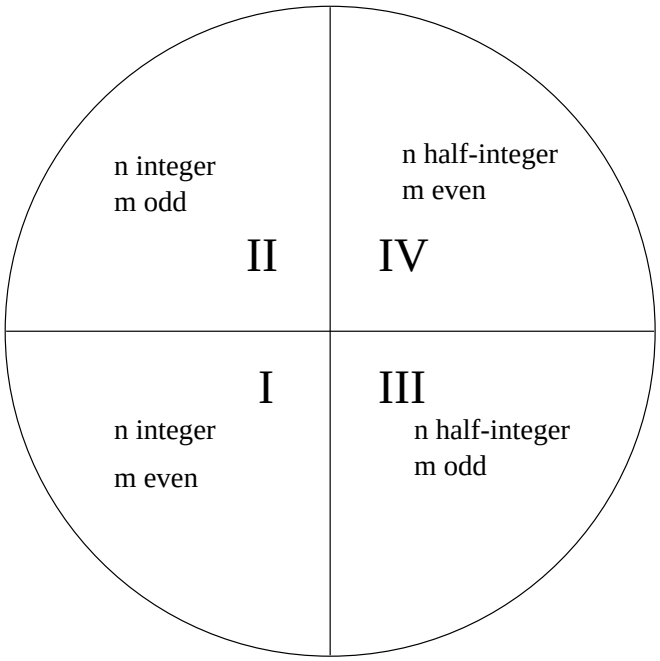


Figure 3: The family of vertex operators $V_{(n,m)}$ with $n \in \mathbb{Z}/2$ and $m \in \mathbb{Z}$. Sector **I** is the common subspace. **I** and **III** are $\mathcal{A}_f$, that defines the UV behaviour of massive Thirring; **I** and **II** are $\mathcal{A}_b$ that is the UV of sine-Gordon, **IV** is a sector of non mutually local vertex operators.

5 Light-cone lattice regularization of Sine-Gordon theory

In this section we present a lattice regularization of the Sine-Gordon model which is particularly suitable to study FSE. It is well known to lattice theorists that there are many actions on a lattice that all discretize the same continuum theory. This means that there are many possible regularizations of the same theory. Lattice researchers are used to choose the action on the lattice possessing the properties that best fit their calculational needs. In the present context what we would like to keep on a lattice discretization of Sine-Gordon model is of course the property of integrability. The light-cone lattice construction we are going to illustrate is a way (not the unique!) to achieve this goal.

5.1 Kinematics on light-cone lattice

It is a usual way to regularize quantum field theories by defining them on a space-like "Hamiltonian" lattice (where time is continuous and space discrete) or space and time-like "Euclidean" lattice (when both space and time are discrete). In statistical mechanics this is not just a regularization method but can be a right microscopic way to describe physical systems. In two dimensions, the most known approach is to define a rectangular lattice with axis corresponding to space and time directions and associate to each site an

interaction depending only on the nearest neighboring sites. In this case the partition function can be expressed in terms of a transfer matrix.

In what follows, a different approach [22] is adopted: Minkowski and Euclidean space-time can, in fact, be discretized along light-cone directions. Light-cone coordinates are:

$$x_{\pm} = x \pm t$$

and the choice

$$\mathcal{M} = \{x_{\pm} = \frac{a}{\sqrt{2}} n_{\pm}, \quad n_{\pm} \in \mathbb{Z}\}$$

defines a light-cone lattice of "events" as in figure 4 (a) . They are spaced by a in the space and time directions and by $a/\sqrt{2}$ in light-cone directions. At every event $P \in \mathcal{M}$ a double light-cone (in the past and in the future) is associated and only events within this light-cone can be causally connected (see fig. 4 (b)). Then, any rational and not greater than 1 speed is permitted for particles, in an infinite lattice. The shortest displacement of the particle (one lattice spacing) is realized at speed ± 1 and corresponds, from the statistical point of view, to nearest neighbors interactions. Smaller speeds can be obtained with displacements longer than the fundamental plaquette, and correspond to high order neighbors interactions. In quantum field theory, these are nonlocal interactions. In the

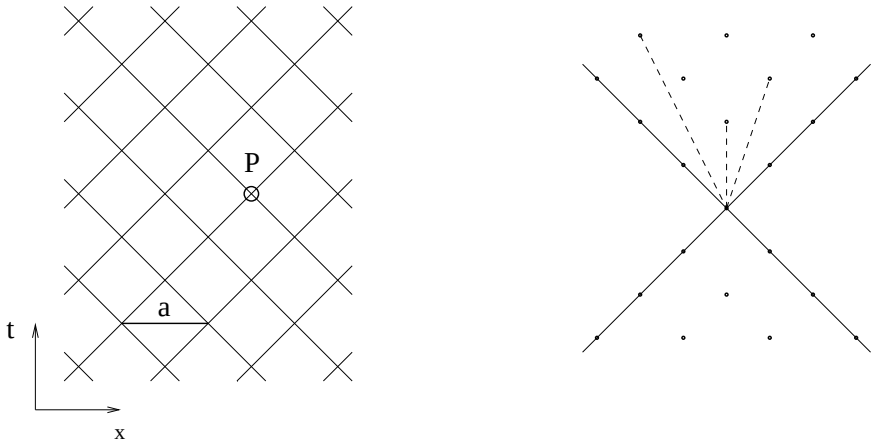


Figure 4: (a) light-cone lattice; (b) double light-cone emanating from a point and high-order interactions (dashed)

following, only the local case (nearest neighbors) is treated. The nearest neighbors of the event P are the four nearest points in the light-cone directions. This implies that particles can have only the speed of light ± 1 and are massless. They are called right-movers (R) and left-movers (L).

In this case, it is possible to introduce a useful language for connection with statistical mechanics associating a particle to a link. Consider the two links in the future that come

out from an event P. Particles R and L in P, by definition, are respectively associated to the right-oriented link and to the left-oriented link. In this way, the state of a link is defined to be the state of the point where it begins (also the opposite choice, of connecting a link with the site where it ends, can be done; it is simply a matter of convention). For example, if in a point O there is a particle R, one tells that the "right-oriented" link outcoming from it is occupied by R. This correspondence of points and links is possible because only local interactions are assumed, and it is useful because the counting of states is simpler. In the following, the lattice is assumed of a finite extent $L = aN$ in space direction (N is the number of sites, counted in the space direction), with periodic boundary conditions, but infinite in time direction. In this way a cylinder topology is defined for space time. The Hilbert space of states in an event P is the tensor product

$$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$$

of R and L space of states. The fact that particles can be classified in left and right does not mean, in general, that the two dynamics are independent, as it happens in CFT.

Call $|\alpha_{Li}, \alpha_{Ri}\rangle$ the generic vector of a basis of $\mathcal{H}_i$ where $i = 1, ..., N$ labels the sites. The notation

$$|\alpha_{2i-1}, \alpha_{2i}\rangle = |\alpha_{Li}, \alpha_{Ri}\rangle$$

is useful and not ambiguous (even number refers to right, odd number refers to left). The total Hilbert space is:

$$\mathcal{H}_N = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

and a basic vector can be represented by

$$|\alpha_1, \alpha_2\rangle \otimes ... \otimes |\alpha_{2N-1}, \alpha_{2N}\rangle = |\alpha_1, \alpha_2, ..., \alpha_{2N}\rangle \in \mathcal{H}_N.$$

Note that in a N sites lattice, due to light-cone, $2N$ labels are required. The state $|\alpha_1, \alpha_2, ..., \alpha_{2N}, t\rangle$ is better thought to lie on links separating two lines of events. The time slices t, $t + \frac{a}{2}$, $t + a$ are taken as represented in fig.5. From this figure it is clear that two different types of evolution operators can be defined, depending on the initial state:

$$\begin{aligned} U_+ |\alpha_1, \alpha_2, ..., \alpha_{2N}, t\rangle &= |\alpha'_1, \alpha'_2, ..., \alpha'_{2N}, t + a/2\rangle \\ U_- |\alpha'_1, \alpha'_2, ..., \alpha'_{2N}, t + a/2\rangle &= |\alpha''_1, \alpha''_2, ..., \alpha''_{2N}, t + a\rangle \end{aligned} \tag{38}$$

where the initial states are chosen as in figure 5. They can be pictorially represented by the links that have to be added to the time slice t to get $t + \frac{a}{2}$ (for U_+) and to $t + \frac{a}{2}$ to get $t + a$ (for U_-)

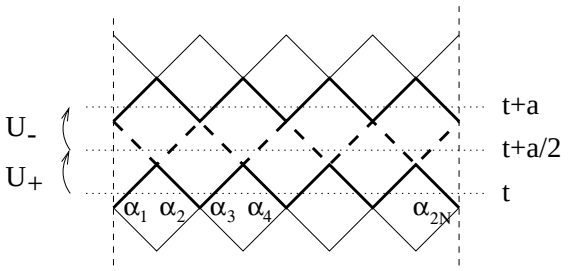
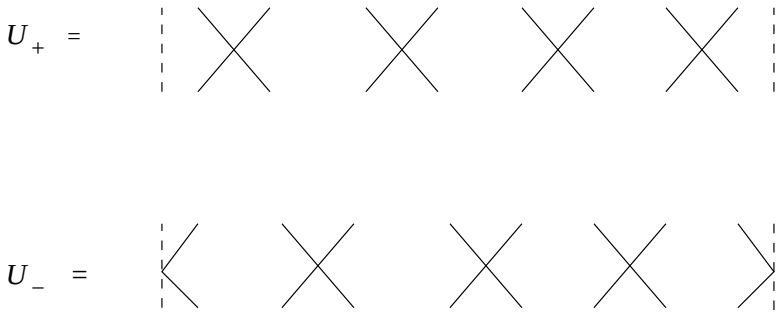


Figure 5: Partial evolution operators. The convention used here for the labeling of states will be used also in the following.



Schrödinger form of equations of motion is used, for a state in Hilbert space.[1] The global time operator can be chosen as

$$U = U_+ U_- \quad \text{or} \quad U' = U_- U_+$$

depending on the initial state. For a consistent quantum theory, both these operators must be unitary. This is guaranteed if the assumption $U_+^\dagger U_+ = U_-^\dagger U_- = 1$ is made, that is the elementary operators themselves must be unitary.

Another operator plays an important role and is defined as follows (the states are at a certain fixed time):

$$V\,|\alpha_1, \alpha_2, ..., \alpha_{2N}, t\rangle = |\alpha_{2N}, \alpha_1, ..., \alpha_{2N-1}, t\rangle \tag{39}$$

It corresponds to an half-space shift in the space direction, with exchange of right and left states (see the figure 6).

Two applications of V give a shift by an entire lattice spacing, then V^2 is the lattice space evolution operator. Also V is a unitary operator. By using the pictorial representations introduced above it is easy to convince oneself that the following relations hold

$$\left[V^2, U_\pm\right] = 0; \qquad U_\pm = V U_\mp V^\dagger. \tag{40}$$

For example, start by taking U_-. To obtain VU_- one has to apply V *after* U_-, which means shift the upper endpoints of U_- by $\frac{a}{2}$ towards right.

[1] In Schrodinger form, if $|\alpha\rangle$ is a state vector, its time evolution is given by $|\alpha, t\rangle = U\,|\alpha, 0\rangle$ where U satisfies motion's equations: $U = Te^{-i\int dtH}$ (Dyson's series)

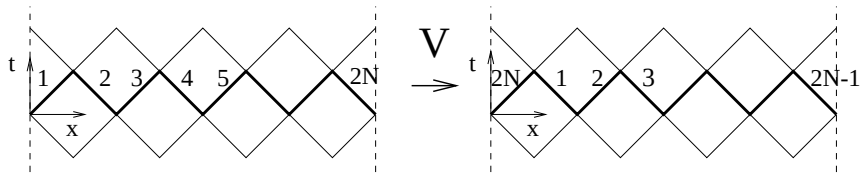
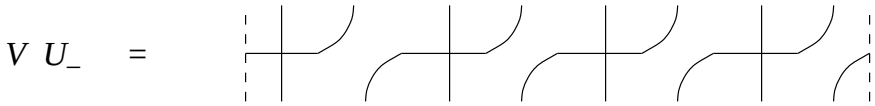


Figure 6: Half-shift operator



Now, by applying $V^\dagger$ to the lower part of this drawing, i.e. shifting the lower endpoints by $\frac{a}{2}$, one can see that $VU_-V^\dagger = U_+$. Analogously, all the other relations can be proven.

As a consequence of (40) the two principal unitary evolution operators, V^2 and U, are commuting:

$$\left[V^2, U\right] = \left[U, V^2\right] = 0.$$

It is natural to identify them as the exponential of the Hamiltonian operator, and the exponential of the linear momentum:

$$\begin{array}{c} U = e^{-iaH} \\ V^2 = e^{-iaP}. \end{array} \tag{41}$$

There are other two important operators, defined as:

$$\begin{array}{c} U_R = U_+ V \\ U_L = U_+ V^\dagger \end{array} \tag{42}$$

They correspond to one step evolution in light-cone directions. They are commuting and give the expressions:

$$\begin{array}{cc} U = U_R U_L & V^2 = U_R U_L^\dagger \\ [U_R, U_L] = 0 & U_R^\dagger U_R = U_L^\dagger U_L = 1 \end{array}$$

then, using also (41) yields:

$$U_R = e^{-i\frac{a}{2}(H+P)}, \qquad U_L = e^{-i\frac{a}{2}(H-P)}. \tag{43}$$

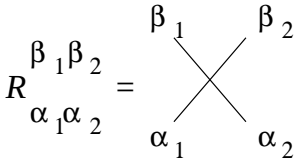
5.2 Dynamics on light-cone lattice

A dynamics can be defined by giving all the amplitudes of the different processes that can take place on the lattice. The fundamental assumption is that at every site a whole process can happen, in the sense that if $|\alpha_L, \alpha_R\rangle_{in}$ and $|\beta_L, \beta_R\rangle_{out}$ are the incoming and outgoing states in a certain site, they can be considered asymptotic states and the transition

amplitude is a (bare) S-matrix element. To each site of the lattice, which is the crossing of 2 incoming and 2 outgoing lines, we associate a matrix given by the Boltzmann weights of a vertex model

$$_{out}\langle \beta_R, \beta_L \,|\alpha_R, \alpha_L\rangle_{in} = R^{\beta_R,\beta_L}_{\alpha_R,\alpha_L}(\lambda) \tag{44}$$

The parameter λ is the so called *spectral parameter* of the R matrix of vertex models [23]. The R matrix can be pictorially represented by drawing the crossing where it is put and labelling the external lines with the 4 indices of the matrix



The system is then defined via its microscopic amplitudes, formally identified with the Boltzmann weights of some lattice model. If the lattice model is integrable, its Boltzmann weights, in the form of R matrix, satisfy Yang-Baxter equations, and our bare scattering theory also does. This is the way integrability is implemented into the lattice definition of the model.

From the definition (38) of evolution operators one can see that they are realized in this context as products of R matrices

$$\langle \alpha_1, \alpha_2, ..., \alpha_{2N}|\, U_+ \,|\beta_1, \beta_2, ..., \beta_{2N}\rangle = \prod_{i=1}^{N} R^{\beta_{2i-1},\beta_{2i}}_{\alpha_{2i-1},\alpha_{2i}}(\lambda - \lambda_i) \tag{45}$$

which, in pictorial form, is nothing else than the already seen definition of U_+, decorated with indices. The operator U_- can be obtained from (40). The product is on all the sites at a given time. The particular choice of the spectral parameter, with a "correction" (inhomogeneity) depending on the site will become clear in a moment.

Apparently, this "phenomenological" approach is quite unusual, because in traditional lattice quantum field theory every site is associated with an interaction potential (ex: $\phi^4(i)$ is the potential on the site i), not a whole scattering process. This can appear as a sort of "macroscopic" approach, not based on fundamental interactions. But the properties of factorizable scattering must be taken into account. Factorization of generic amplitudes in $2 \to 2$ particles amplitudes is a sort of quantum superposition principle and the remarkable fact is that between one scattering and the other, the particles are asymptotic ones, that means that they are free. Every point contains all the interaction. This is what was assumed in the definition of the lattice. Then it is perfectly justified that every site is connected with a whole $2 \to 2$ particle scattering process.[2]

[2]In the next paragraphs, it will be shown that, for the particular case of the 6 vertex R matrix, a more traditional lattice QFT approach can be formulated, in terms of a fermionic field.

5.3 Euclidean transfer matrix

In order to relate the evolution operators to something computable, we have to consider the Euclidean lattice vertex model transfer matrix. Consider a two dimensional Euclidean square lattice, with periodic boundary conditions, in both the directions. The links are the physical objects of the system. They can be in different states belonging to the vector spaces $\mathcal{A}$ (horizontal) and $\mathcal{V}$ (vertical). In principle, such vector spaces can be different, but in the following they will be assumed as isomorphic. At every site we associate a Boltzmann weight depending on the four links crossing at this site and on a spectral parameter λ (see fig. 7). The simplest case is to take the same Boltzmann weights at all sites, but more

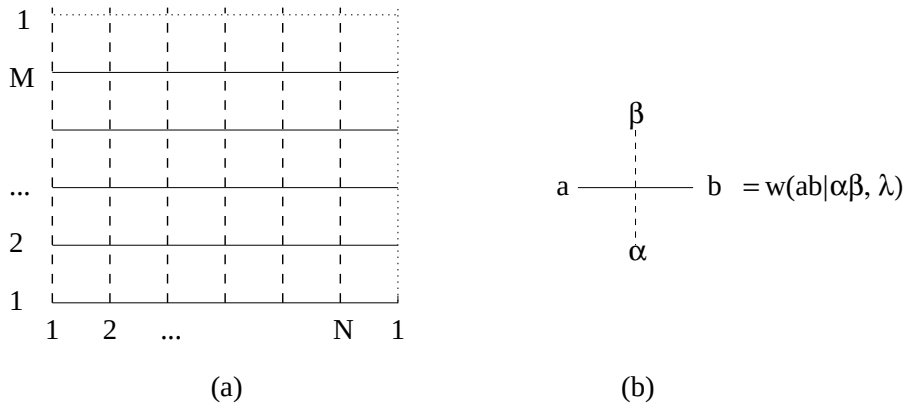
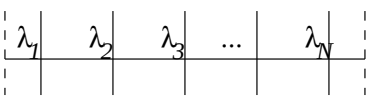


Figure 7: (a) periodic (toroidal) square lattice; (b) Boltzmann weight for a site; the labels are: $a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathcal{V}$

general configurations are possible. For our purposes it will be interesting to consider the case with an inhomogeneity λ_i at each site, where i is the column index (i.e. all the sites on a column have the same inhomogeneity), in the sense that the Boltzmann weight in the column i is taken to be $w(ab|\alpha\beta,\, \lambda - \lambda_i) = R^{\beta\, b}_{a,\alpha}(\lambda - \lambda_i)$. Also for the boundary conditions it is possible to assume more general configurations than the simplest one (i.e. toroidal b.c.). Assume that between the column N and the N+1 (that is 1) there is a nontrivial seam line, in such a way that the Boltzmann weights on the column N (with respect to the normal ones) are given by $e^{i\omega b} R^{\beta\, b}_{a,\alpha}(\lambda - \lambda_N)$. This choice is made because only the link b crosses the seam line. It implements the so called *twisted boundary conditions* (of course, $\omega = 0$ reproduces the periodic case).

The object of interest for us is the row-to-row transfer matrix. It is a product of concatenated R matrices along a whole horizontal line of the lattice and describes the evolution from one time slice to the next in Euclidean lattice. For simplicity, let us insist on the pictorial notation

$t\,(\,\lambda\,|\,\{\lambda_{\,j}\}\,) \quad =$



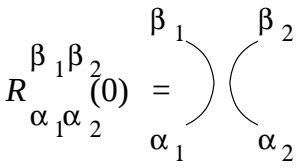
We have omitted the indices on the two rows to avoid useless heaviness of notation.

This object is well known in the statistical mechanics literature and for many integrable models it can be diagonalized exactly by Bethe Ansatz methods, as we shall illustrate in next sections. Unfortunately, it refers to a row-to-row evolution on an Euclidean lattice, and not to the case we are interested in of an evolution in tine in our light-cone lattice. Is there a way to relate this known object to the operators $U_{\pm}$ or $U_{L,R}$ introduced above?

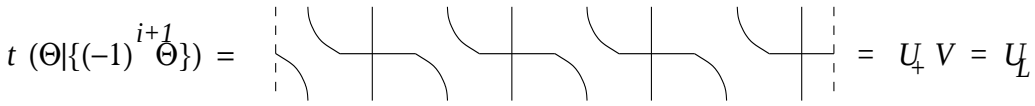
The trick comes eventually by considering a particular distribution of inhomogeneities. Compute the transfer matrix at a value Θ of the spectral parameter, and choose inhomogeneities to be $\lambda_i = (-1)^{i+1}\Theta$. In this case, all the R matrices at odd sites are calculated at spectral parameter 2Θ, while those at even sites are calculated at spectral parameter 0. The nice fact is that, for all unitary R matrices

$$R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}(0) = \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}$$

or in graphical form



The transfer matrix $t(\Theta|\{(-1)^{i+1}\Theta\})$ then turns out to be equivalent to the matrix element of the operator U_+ taken between two time slices of the light-cone lattice, as it appears clearly from this pictorial representation



Analogously the operator U_R is realized by $t^{-1}(\Theta|\{(-1)^{i+1}\Theta\})$. Diagonalization of t gives therefore the eigenvalues of U_R and U_L which are the exponentials of $E + P$ and $E - P$ respectively. If we are able to compute the eigenvalues of t, we are able to calculate the energy levels and the total momenta of states in the theory defined on the light-cone lattice.

5.4 6 vertex model: main results

The theory developed up to now is general and not referred to a specific model. The simplest non trivial case is the 4×4 R matrix, corresponding to the choice $\mathcal{A} = \mathcal{V}$. As shown in [23], the most general solution is the so called 8 vertex model. This name means that only 8, between the 16 entries of the R matrix, are nonzero. A special case is the 6 vertex model, for which many results have been obtained in the light-cone description: this will be the principal object in the following. The R matrix has the form (lower index

are rows and upper index are columns)

$$R(\vartheta,\gamma)=\begin{pmatrix} a & & & \\ & c & b & \\ & b & c & \\ & & & a \end{pmatrix} \tag{46}$$

There is a well known mapping between vertex models and spin chains (see [23]), i.e. the transfer matrix is the exponential of the quantum Hamiltonian of the chain:

$$t^{(N)}=e^{-H}.$$

In the case of 8 vertex model, the Hamiltonian is the XYZ(1/2) chain, while in the special case of 6 vertex, is the XXZ(1/2) chain. In what follows, this identification can be useful to interpret some facts connected with Bethe Ansatz. The XXZ(1/2) chain Hamiltonian is given by:

$$H=\sum_{i=1}^{N}\left[\sigma_x^i\sigma_x^{i+1}+\sigma_y^i\sigma_y^{i+1}+(1-\cos\gamma)\sigma_z^i\sigma_z^{i+1}\right]$$

and γ is the anisotropy. The σ are Pauli matrices. The total z-component of the spin, which is a conserved quantity in this system, will play an important role in Bethe Ansatz.

It is possible to give a particle interpretation to the same R matrix (46) on the light-cone lattice. The simplest case of particles obeying Pauli exclusion principle (fermions) and without internal degrees of freedom (color number) is assumed. This means that in an event only one particle of type R and one L at most can take place. In other words, one link has two states: empty or occupied. At every point there are four links, that means 16 possible configurations associated to it. In terms of events, these are the possible configurations that connect a point with the nearest neighbors in the future.

Assume now that only amplitudes that conserve the total number of particles (R+L) are non vanishing. This reduces to 6 the permitted configurations, as it is shown in the figure 8. This is simply the 6 vertex model whose R matrix is written in (46). The assumption

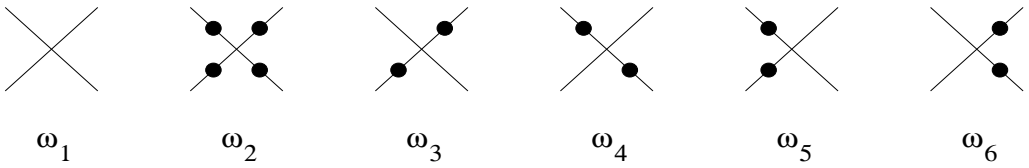


Figure 8: 6 permitted amplitudes. Dots are particles.

of integrability for this amplitudes gives the general six-vertex model. The requirement of symmetry under parity transformation implies that $\omega_3=\omega_4$ and $\omega_5=\omega_6$. The convention adopted in (46) gives

$$b=\omega_3=\omega_4,\quad c=\omega_5=\omega_6.$$

This R matrix can be written in an operatorial form, by defining a lattice chiral fermion $\psi_{A,n}$, with $A = R, L$, and n labels the sites. The anticommutation rules are the canonical ones:

$$\{\psi_{A,n}, \psi_{B,m}\} = 0, \qquad \left\{\psi_{A,n}, \psi^\dagger_{B,m}\right\} = \delta_{AB}\delta_{nm}. \tag{47}$$

This fermion has some interesting properties, that are exposed in the paper [22], and are sketched in the following list:

1. the R matrix and all the other operators U_{any} can be written in an operatorial form in terms of the fermion;

2. the lattice Hamiltonian, in the free case $\omega_1 = \omega_2 = b = 1,\ c = 0$, can be explicitly written; by this, the following dispersion relation can be obtained: $E = \pm k$; this is the dispersion relation for a free massless particle; the unusual fact is that it is monotonous, then there is no doubling of fermions; this is a consequence of the non-locality of the Hamiltonian

3. the lattice Hamiltonian admits a continuum limit $N \to \infty$ and $a \to 0$ but with $L = Na$ fixed; locality is recovered in this limit; the continuum equations of motion are those of the massive Thirring model. The space is compactified on a cylinder.

In the continuum limit, the massive Thirring model emerges as the field theory characterizing the scaling behavior of the dynamics on the lattice (remember that L is finite; the scaling behaviour is understood in terms of this L). We conclude that the theory we have constructed is a lattice regularization of the Thirring model suitable to study its integrability properties.

5.5 6 vertex model: Bethe Ansatz

The assumptions of unitarity and hermitian analyticity will be taken into account, for the R matrix, and this requires that the variables in (46) must have the specific form:

$$a = a(\vartheta, \gamma) = \sinh(i\gamma - \vartheta), \quad b = b(\vartheta, \gamma) = \sinh\vartheta, \quad c = c(\vartheta, \gamma) = i\sin\gamma \tag{48}$$

The transfer matrix defined by this R matrix can be diagonalized with Bethe Ansatz method [37]. In terms of the spin chain, this means that there are two operators, usually indicated by $B(\vartheta)$ and $C(\vartheta)$, whose expression is known, and there is a "reference state"[3] $|\Omega\rangle$ such that:

$$C(\vartheta)\,|\Omega\rangle = 0$$

[3]The "reference state" here is only a mathematical object. Physically speaking, it corresponds to the ferromagnetic state with all the spins up.

and

$$B(\vartheta_1)...B(\vartheta_M)\left|\Omega\right\rangle, \tag{49}$$

for appropriate values of ϑ_j, is an eigenstate of the transfer matrix. The "appropriate values" of ϑ_j can be obtained as the solution of a set of M coupled nonlinear equations, that are called Bethe Ansatz equations. In general, the transfer matrix contains all the conserved charges, in particular the Hamiltonian. Then the Bethe Ansatz eigenstates are also eigenstates of the Hamiltonian. The Hilbert space of the theory and the action of conserved charges on it are then exactly known.

All this computations for the 6 vertex model were performed in [23, 24, 25]; the final results are written here for the eigenvalues $\tau(\vartheta,\Theta,\omega)$ of the inhomogeneous and twisted transfer matrix (here ϑ is the spectral parameter, Θ the inhomogeneity and ω the twist)

$$\tau(\vartheta,\Theta,\omega)=e^{i\omega}\left[a(\vartheta-\Theta)\,a(\vartheta+\Theta)\right]^N\prod_{j=1}^{M}\frac{\sinh\frac{\gamma}{\pi}\left[i\frac{\pi}{2}+\vartheta_j+\vartheta\right]}{\sinh\frac{\gamma}{\pi}\left[i\frac{\pi}{2}-\vartheta_j-\vartheta\right]}+$$
$$+e^{-i\omega}\left[b(\vartheta-\Theta)\,b(\vartheta+\Theta)\right]^N\prod_{j=1}^{M}\frac{\sinh\frac{\gamma}{\pi}\left[i\frac{3\pi}{2}-\vartheta_j-\vartheta\right]}{\sinh\frac{\gamma}{\pi}\left[-i\frac{\pi}{2}+\vartheta_j+\vartheta\right]}$$

and the values of ϑ_j are defined by the set of coupled nonlinear equations called Bethe Ansatz equations:

$$\left(\frac{\sinh\frac{\gamma}{\pi}\left[\vartheta_j+\Theta+\frac{i\pi}{2}\right]\sinh\frac{\gamma}{\pi}\left[\vartheta_j-\Theta+\frac{i\pi}{2}\right]}{\sinh\frac{\gamma}{\pi}\left[\vartheta_j+\Theta-\frac{i\pi}{2}\right]\sinh\frac{\gamma}{\pi}\left[\vartheta_j-\Theta-\frac{i\pi}{2}\right]}\right)^N=$$
$$=-e^{2i\omega}\prod_{k=1}^{M}\frac{\sinh\frac{\gamma}{\pi}\left[\vartheta_j-\vartheta_k+i\pi\right]}{\sinh\frac{\gamma}{\pi}\left[\vartheta_j-\vartheta_k-i\pi\right]}$$

where $2N$ is the length of the chain and N the number of sites in a row of the light-cone lattice. The ϑ_j are called *Bethe roots*, and in principle, can take any complex value. But there is a periodicity in their imaginary part:

$$\vartheta_j\;\rightarrow\;\vartheta_j+\frac{\pi^2}{\gamma}i \tag{50}$$

then only a strip around the real axis must be taken into account for the Bethe roots:

$$\vartheta_j\in\mathbb{R}\times i\;\left]-\frac{\pi^2}{2\gamma},\frac{\pi^2}{2\gamma}\right]. \tag{51}$$

Moreover, only the range $0<\gamma<\pi$ will be examined.

The whole spectrum of the theory can be obtained using all the Bethe configurations having $M\leq N$ and $\vartheta_j\neq\vartheta_k$ for every $j\neq k$. In general for a state with M roots the third component of the spin of the chain is

$$S=N-M, \tag{52}$$

because every operator $B(\vartheta_i)$ counts as -1 spin.

This XXZ(1/2) chain has 2 states in every site, then 2^{2N} states in total. Then the energy spectrum is upper and lower bounded. Changing the sign of the Hamiltonian gives another permitted physical system. This means that there are two possible vacua. The first one is the so called ferromagnetic ground state, corresponding to $M = 0$ that is the reference state $|\Omega\rangle$. It has spin $S = N$. The second one is the antiferromagnetic ground state, that can be obtained with $M = N$ and all the roots ϑ_i real. It has spin $S = 0$.

In what follows, only the antiferromagnetic ground state will be considered, because it has one important property: in the thermodynamic limit ($N \to \infty$) it can be interpreted as a Dirac vacuum (a sea of particles created by B) and the excitations on this vacuum behave as particles.

The energy E and momentum P of a state can be read out by the transfer matrix eigenvalues. The final form is:

$$e^{i\frac{a}{2}(E\pm P)} = e^{\pm i\omega} \prod_{j=1}^{M} \frac{\sinh\frac{\gamma}{\pi}\left[i\frac{\pi}{2} - \Theta \pm \vartheta_j\right]}{\sinh\frac{\gamma}{\pi}\left[i\frac{\pi}{2} + \Theta \mp \vartheta_j\right]}. \tag{53}$$

Other integrals of motion can be obtained in a similar way by transfer matrix. Note that the second term in the transfer matrix expression vanishes because $b(0) = 0$.

6 Nonlinear Integral Equation from Bethe Ansatz

In this section the fundamental nonlinear integral equation driving sG scaling functions is derived. In the literature it is known as Destri-de Vega equation. It was obtained first in [26, 35] for the vacuum scaling function. The treatment of excited states was pioneered in [38] and refined in [27, 39], to arrive to the final form, in [28, 30].

6.1 Counting function

It is possible to write the Bethe equations (3) in terms of a *counting function* $Z_N(\vartheta)$.

First, introduce the function[4]

$$\phi_\nu(\vartheta) = i\log\frac{\sinh\frac{1}{p+1}(i\pi\nu + \vartheta)}{\sinh\frac{1}{p+1}(i\pi\nu - \vartheta)}, \qquad \phi_\nu(-\vartheta) = -\phi_\nu(\vartheta)$$

The oddness on the analyticity strip around the real axis defines a precise choice of the logarithmic branch. The counting function is defined by

$$Z_N(\vartheta) = N[\phi_{1/2}(\vartheta + \Theta) + \phi_{1/2}(\vartheta - \Theta)] - \sum_{k=1}^{M} \phi_1(\vartheta - \vartheta_k) + 2\omega \tag{54}$$

[4]In the following we use the parameter $p = \frac{\pi}{\gamma} - 1$. It will later turn out to be exactly the same as the one introduced in (37).

The logarithm of the Bethe equations boils down to the simple condition

$$Z_N(\vartheta_j) = 2\pi I_j\,, \quad I_j \in \mathbb{Z} + \frac{1+\delta}{2}, \quad \delta = (M)_{mod\,2} = (N-S)_{mod\,2} \in \{0,1\} \tag{55}$$

The number I_j plays the role of a quantum number for the Bethe basic vectors (49) and it must be chosen depending on the value of δ. Notice that δ and ω play a similar role, because both produce a shift in the quantum numbers I_j (if ω is absorbed in the definition of I_j): in the first case the shift is exactly π, in the second case it is a real (possibly complex) number. This means that the variable δ can be absorbed in ω but the most convenient choice is to use them both.

Observe that Bethe roots can be obtained as zeros of the equation:

$$1 + (-1)^{\delta} e^{iZ_N(\vartheta_j)} = 0 \tag{56}$$

6.2 Classification of Bethe roots and counting equation

From Bethe Ansatz it is known that a Bethe state (49) is uniquely characterized by the set of quantum numbers $\{I_j\}_{j=1,\dots,M}$, $M \leq N$ that appear in (55). Notice that $M \leq N$ means $S \geq 0$. The values of ϑ_j to put in (49) can be obtained solving Bethe equations. It is also known that only states with

$$\vartheta_j \neq \vartheta_i, \quad \forall\, j \neq i$$

are allowed. It is a sort of fermionic character for Bethe states [29].

Bethe roots can either be real or appear in complex conjugate pairs. In the specific case (3), there is another possibility, due to periodicity (50): if a complex solution has imaginary part $Im\,\vartheta = \frac{\pi}{2}(p+1)$ it appears as a single root (in (3) it produces the left hand side real, then its complex conjugate is not required). It is called *self-conjugate root*. Remember now that the maximal number of real roots ($M = N$) describes the antiferromagnetic ground state. From the point of view of the counting function, a more precise classification of roots is required:

- *real roots*; they are real solutions of (55) that appear in the vector (49); their number is M_R;
- *holes*; real solutions of (55) that do NOT appear in the vector (49); their number is N_H;
- *special roots/holes* (special objects); they are real roots or holes where the derivative $Z'_N(\vartheta_j)$ is negative[5]; their number is N_S; they must be counted both as "normal" and as "special" objects;
- *close pairs*; complex conjugate solutions with imaginary part in the range $0 < |Im\,\vartheta| < \pi \min(1,p)$; their number is M_C;

[5]The characteristics of this type of solutions will be clarified in the next sections.

- *wide roots in pairs*; complex conjugate solutions with imaginary part in the range $\pi \min(1,p) < |Im\,\vartheta| < \pi\frac{p+1}{2}$;
- *self-conjugate roots* (wide roots appearing as single); $Im\,\vartheta = \pi\frac{p+1}{2}$; their number is M_{SC}.

The total number of wide roots appearing in pairs or as single is M_W. The following notation will be used (sometimes) for later convenience, to indicate the position of the solutions: h_j for holes, y_j for special objects, c_j for close roots, w_j for wide roots.

Complex roots with imaginary part larger than the self-conjugates are not required because of the periodicity of Bethe equations. A graphical representation of the various types of solutions is given in figure 9.

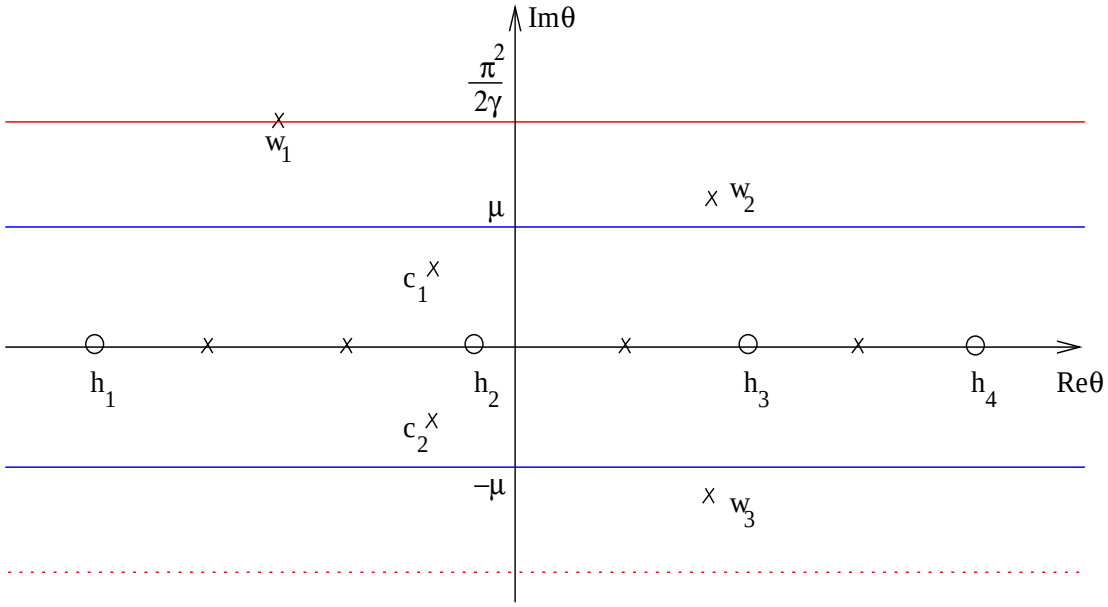


Figure 9: The different types of roots and holes and their position in the complex plane. μ denotes $\pi \min(1,p)$. The upper line at $\frac{\pi^2}{2\gamma} = \frac{\pi}{2}(p+1)$ is the self-conjugate one.

An important remark must be done: from the definition of Z_N (54) it is obvious that only for states without complex roots the fundamental strip for $\phi_\nu(\vartheta))$, that is the largest strip around the real axis without singularities, is the fundamental strip for Z_N. In all the other cases the analyticity strip for Z_N is narrower, and depends on the imaginary parts of the complex roots.

An important property follows from this classification: the Z_N function is *real analytic* if ω is a real number

$$Z_N\left(\vartheta^*\right) = \left(Z_N(\vartheta)\right)^* \tag{57}$$

By considering asymptotic values of $\phi_\nu(\vartheta)$ and $Z_N(\vartheta)$ for $\vartheta \to \pm\infty$, it is possible to obtain an equation relating the numbers of all the various types of roots. We refer the

reader interested in the details of the derivation to [27, 30]. Here we only mention the final result, in the form where the continuum limit $N \to \infty$, $a \to 0$ and $L = Na$ finite, is already taken

$$N_H - 2N_S = 2S + M_C + 2\,\theta(p-1)\,M_W \tag{58}$$

where $\theta(x)$ is the step function: $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$. Recall that S is a nonnegative integer. In the case of $\omega = 0$, it turns out that N_H is even (M_C is the number of close roots, and is even).

The most important fact is that the number of real roots does not appear in this equation . This fact, together to what will be explained in the next paragraph, allows to consider the real roots as a sea of particles (Dirac vacuum) and all other types of solutions (holes, complex) as excitations on this sea.

6.3 Non linear integral equation

Let $\hat{x}$ be a real solution of the Bethe equation. Thanks to Cauchy theorem, an analytic function $f(x)$ on an appropriate strip containing the real axis admits the following representation

$$f(\hat{x}) = \oint_{\Gamma_{\hat{x}}} \frac{d\mu}{2\pi i} \frac{f(\mu)}{\mu - \hat{x}} = \oint_{\Gamma_{\hat{x}}} \frac{d\mu}{2\pi i} f(\mu) \frac{(-1)^\delta e^{iZ_N(\mu)} i Z_N'(\mu)}{1 + (-1)^\delta e^{iZ_N(\mu)}} \tag{59}$$

where $\Gamma_{\hat{x}}$ is a anti-clockwise curve encircling $\hat{x}$ and avoiding other singularities of the denominator, i.e. other Bethe solutions (real or complex). In the region where $\phi_1(\vartheta)$ is analytic, we can use (59) to write

$$\begin{aligned} \sum_{k=1}^{M_R+N_H} \phi_1(\vartheta - x_k) &= \sum_{k=1}^{M_R+N_H} \oint_{\Gamma_{x_k}} \frac{d\mu}{2\pi i} \phi_1(\vartheta - \mu) \frac{(-1)^\delta e^{iZ_N(\mu)} i Z_N'(\mu)}{1 + (-1)^\delta e^{iZ_N(\mu)}} = \\ &= \oint_{\Gamma} \frac{d\mu}{2\pi i} \phi_1(\vartheta - \mu) \frac{(-1)^\delta e^{iZ_N(\mu)} i Z_N'(\mu)}{1 + (-1)^\delta e^{iZ_N(\mu)}} \end{aligned} \tag{60}$$

The sum on the contours was modified to a unique curve Γ encircling all the real solutions x_k, and avoiding the complex Bethe solutions (this is possible because they are finite in number), as in the figure 10.

Clearly the Γ curve must be contained in the strip

$$0 < \eta_+, \eta_- < \min\{\pi, \pi p, |Im\, c_k| \;\forall\, k\}$$

Without loss of generality, assume that $\eta_+ = \eta_- = \eta$, and deform Γ to the contour of the strip characterized by η. The regions at $\pm\infty$ do no contribute because of the vanishing of ϕ', then the integral can be performed on the lines $\mu = x \pm i\eta$, where x is real. After

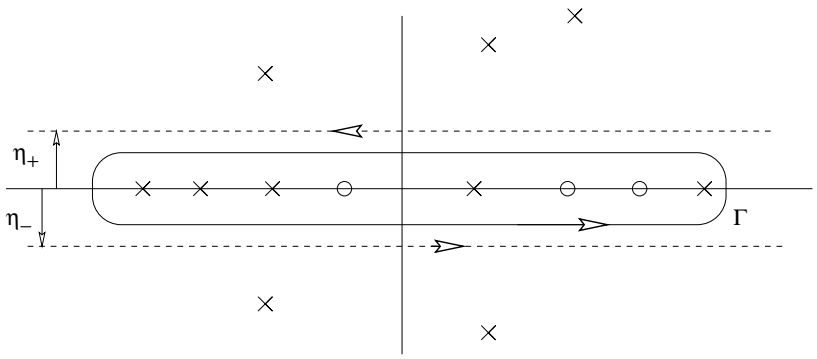


Figure 10: Contours for the integration. The crosses are roots while the circles are holes.

algebraic manipulations involving integrations by parts and convolutions (for details see [30]) one arrives at a non-linear integral equation (NLIE) to be satisfied by $Z_N(\vartheta)$

$$\begin{aligned} Z_N(\vartheta) &= 2N \arctan \frac{\sinh \vartheta}{\cosh \Theta} + \sum_{k=1}^{N_H} \chi(\vartheta - h_k) - 2\sum_{k=1}^{N_S} \chi(\vartheta - y_k) - \\ &- \sum_{k=1}^{M_C} \chi(\vartheta - c_k) - \sum_{k=1}^{M_W} \chi(\vartheta - w_k)_{II} + \\ &+ 2Im \int_{-\infty}^{\infty} d\rho G(\vartheta - \rho - i\eta) \log\left(1 + (-1)^{\delta} e^{iZ_N(\rho + i\eta)}\right) \end{aligned}$$

The kernel

$$G(\vartheta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{ik\vartheta} \frac{\sinh \frac{\pi(p-1)k}{2}}{2 \sinh \frac{\pi p k}{2} \cosh \frac{\pi k}{2}}\,,\; p = \frac{\pi}{\gamma} - 1 \tag{61}$$

presents a singularity at the same place where $\phi_1(\vartheta)$ does. Any analytic continuation outside the fundamental strip $0 < |Im\,\vartheta| < \pi \min(1, p)$ (I determination region) must take this fact into account. The source terms are given by

$$\chi(\vartheta) = 2\pi \int_0^{\vartheta} dx G(x)$$

and

$$\chi(\vartheta)_{II} = \begin{cases} \chi(\vartheta) + \chi\left(\vartheta - i\pi \mathrm{sign}\left(Im\vartheta\right)\right)\,, & p > 1\,, \\ \chi(\vartheta) - \chi\left(\vartheta - ip\pi \mathrm{sign}\left(Im\vartheta\right)\right)\,, & p < 1\,. \end{cases}$$

is a modification of the source term due to the analytic continuation over the strip $0 < |Im\,\vartheta| < \pi \min(1, p)$, i.e. in the so called II determination region.

Such equation is equivalent to the Bethe Ansatz, in the sense that, once solved, it gives the counting function from which the Bethe roots can be reconstructed. Once the Bethe roots are known, one can use them into eqs.(53) to compute the energy and momentum for a given state.

6.4 Continuum limit

Although such NLIE is already a precious tool for the lattice model itself, its importance becomes essential when a continuum limit is done.

The continuum limit must be performed carefully in the light-cone lattice setup. As already mentioned, one has to take $N \to \infty$ (the so called thermodynamic limit of Statistical mechanics) and the $a \to 0$ limit simultaneously, in such a way that the product $L = Na$ keeps finite. This is the way to implement finite size in the continuum theory, removing all lattice artifacts but without destroying the cylindrical geometry. However, one can convince himself, by performing calculations explicitly, that if the limit is taken by keeping the Θ parameter fixed, the lattice NLIE blows up to infinity and looses meaning. This reflects the fact that the number of roots, and hence of Bethe equations goes as N and becomes infinite in the thermodynamic limit, thus becoming intractable. However, as already forecast in ref.[31], if one assumes a dependence of Θ on N of the form

$$\Theta \approx \log \frac{4N}{\mathcal{M}L}. \tag{62}$$

it is possible to get a finite limit of the lattice NLIE. This limit is exactly the one that brings the lattice fermion introduced in 5.4 into the Thirring fermion on the continuum. It is therefore very natural to give meaning to the continuum NLIE thus obtained as the remnant of Bethe Equations for the renormalized QFT. Notice that sending $\Theta \to \infty$ in this way naturally introduces a renormalized physical mass $\mathcal{M}$.

The *continuum counting function* is defined by:

$$Z(\vartheta) = \lim_{N \to \infty} Z_N(\vartheta). \tag{63}$$

We present here the continuum NLIE in its full generality

$$Z(\vartheta) = l \sinh \vartheta + g(\vartheta|\vartheta_j) + \alpha + 2Im \int_{-\infty}^{\infty} dx G(\vartheta - x - i\eta) \log\left(1 + (-1)^{\delta} e^{iZ(x+i\eta)}\right) \tag{64}$$

where $l = \mathcal{M}L$. The source term $g(\vartheta|\vartheta_j)$ is

$$g(\vartheta|\vartheta_j) = \sum_{k=1}^{N_H} \chi(\vartheta - h_k) - 2\sum_{k=1}^{N_S} \chi(\vartheta - y_k) - \sum_{k=1}^{M_C} \chi(\vartheta - c_k) - \sum_{k=1}^{M_W} \chi(\vartheta - w_k)_{II}$$

The positions of $\{\vartheta_j\} \equiv \{h_j\,,\, y_j\,,\, c_j\,,\, w_j\}$ are fixed by the Bethe quantization conditions

$$Z(\vartheta_j) = 2\pi I_j \quad , \quad I_j = \mathbb{Z} + \frac{1+\delta}{2}$$

The parameter δ can be both 0 or 1 in principle. On the lattice it was determined by the total number of roots, which has become infinite here together with N. We shall see that

locality implements restrictions on δ. The vacuum corresponds to a choice $\delta = 0$. The parameter α takes into account the twist

$$\alpha = \omega \frac{p+1}{p} + \pi \frac{p-1}{2p} \left(\left\lfloor \frac{1}{2} + \frac{S}{p+1} + \frac{\omega}{\pi} \right\rfloor - \left\lfloor \frac{1}{2} + \frac{S}{p+1} - \frac{\omega}{\pi} \right\rfloor \right) \tag{65}$$

In (3) the twist term ω is invariant for the shift

$$\omega \ \rightarrow \ \omega + \pi .$$

The same sort of invariance is required in α for the NLIE (which is equivalent to Bethe equations). It is simple to verify that the expression for α (65) displays the following symmetry

$$\alpha \ \rightarrow \ \alpha + 2\pi \ \text{ when } \ \omega \ \rightarrow \ \omega + \pi \, .$$

Note that shifting α by 2π is an invariance of the NLIE (64), if an appropriate redefinition of the Bethe quantum numbers is made: $I_j \ \rightarrow \ I_j + 1$. This shift does not affect physical quantities, that depend only on the variables ϑ_j. The untwisted case corresponds to $\alpha = 0$.

In practical calculations, one starts from a $Z(\vartheta)$ without any roots other than the real ones (i.e. the counting function of the vacuum). This will be a solution of (64) with $g(\vartheta|\vartheta_j) = 0$. It can be got by numerical iteration starting from the driving term $l \sinh \vartheta$. The numerical convergence is normally quite fast. It is more efficient for large l, while it becomes a bit slower in the region near $l \rightarrow 0$. Once the vacuum $Z(\vartheta)$ is determined, one can start fixing the number of roots, holes, etc... that must appear in a particular state he wants to study. This configuration must, of course, be compatible with the counting equation (58). The first trial positions for $\vartheta_j \in \{h_j \, , \, y_j \, , \, c_j \, , \, w_j\}$ will be determined by the quantization conditions $Z(\vartheta_j) = 2\pi I_j$. The (integers or half-integers) numbers I_j are chosen in principle freely, but it turns out that some of them are constrained in terms of the others. In such a way a first trial of $g(\vartheta|\vartheta_j)$ is created and put into the NLIE. Iteration is redone, in order to get a better approximation for $Z(\vartheta)$, from which a better determination of the positions of ϑ_j can be obtained again by using the quantization rules, which in turn gives a better $g(\vartheta|\vartheta_j)$, and so on so forth, up to convergence. All this procedure can be done in few minutes of computing on a typical Linux/Intel platform without resorting to any supercomputer or other technically advanced tool.

6.5 Energy and momentum expressions

With a procedure analogous to the one sketched above, i.e. apply the residue trick and then do a continuum limit along the lines of the previous section, it is possible to give integral expressions for the energy and momentum. Starting from (53) one can isolate an extensive (i.e. proportional to N) bulk term to be subtracted on the continuum. The remaining finite part of the energy can be written in a form where the positions of the

various kinds of roots and the function $Z(\vartheta)$ are the only inputs

$$\begin{aligned} E - E_{bulk} &= \mathcal{M}\left(\sum_{j=1}^{N_H}\cosh h_j - 2\sum_{j=1}^{N_S}\cosh y_j - \sum_{j=1}^{M_C}\cosh c_j + \sum_{j=1}^{M_W}(\cosh w_j)_{II} - \right. \\ &\left. - \int_{-\infty}^{\infty}\frac{dx}{2\pi}2Im\left[\sinh(x+i\eta)\log(1+(-1)^{\delta}e^{iZ(x+i\eta)})\right]\right) \end{aligned} \tag{66}$$

$$\begin{aligned} P &= \mathcal{M}\left(\sum_{j=1}^{N_H}\sinh h_j - 2\sum_{j=1}^{N_S}\sinh y_j - \sum_{j=1}^{M_C}\sinh c_j + \sum_{j=1}^{M_W}(\sinh w_j)_{II} - \right. \\ &\left. - \int_{-\infty}^{\infty}\frac{dx}{2\pi}2Im\left[\cosh(x+i\eta)\log(1+(-1)^{\delta}e^{iZ(x+i\eta)})\right]\right) \end{aligned} \tag{67}$$

Therefore, once the $Z(\vartheta)$ and the ϑ_j positions are calculated by the numerical iterative procedure illustrated above, it is sufficient to plug their values into the expressions (66) and (67) to get the values of these quantities for the given state with the wanted accuracy. These values are *numerical but exact,* in the sense that, although there is lack of a closed formula for them, their computation is done without any approximation other than the technical ones introduced by the computer truncations.

6.6 Physical interpretation

The limit procedure described in the previous section is mathematically consistent, but the question is if from the physical point of view it describes a consistent quantum theory and allows for a meaningful physical interpretation.

First of all, an important remark must be made about the allowed values for the XXZ spin S. It is clear from (52) that on the lattice only integer and nonnegative values can be taken into account for S. But on the continuum the definition of S is no more related to the Bethe state (that is undefined), instead it is given implicitly by (58). Then, *"a priori"*, there are no arguments that constrain its values to be integers. As shown in [41], the half-integer choice for S is necessary (and gives completely consistent results) to describe odd numbers of particles.

At this point the following physical interpretation can be proposed. It will be refined to describe the correspondence with particles. Also it will be supported by many arguments that will be clarified in the following:

- the physical vacuum (Hamiltonian ground state) corresponds to absence of sources (i.e. holes, complex roots,...); all the sources are excitations on this vacuum. Indeed, it can be shown that they give positive contribution to the energy, behaving as particle excitations on a vacuum state.

- for $\omega = 0$ and at the various values of S this theory describes the sine-Gordon/massive Thirring model on a finite space of size L; $2S$ is the topological charge and can take nonnegative integer values.

- for the values

$$\omega = \frac{k\pi}{s}\,, \qquad k = 1, ..., q' - 1 \tag{68}$$

  it describes the quantum restrictions of sine-Gordon model, i.e. the massive integrable theory obtained perturbing the minimal models $Vir(r, s)$ by the operator $\Phi_{(1,3)}$.

- the real roots have disappeared from the counting in the continuum limit. They actually become infinite in number and are taken into account by the integral term, both in the NLIE and in the energy-momentum expression. They can be interpreted as a sort of Dirac sea on which holes and complex roots build particle excitations. Note that the presence of holes or complex roots, through the source term $g(\vartheta|\vartheta_j)$ distorts the Dirac sea too. This can be interpreted as vacuum polarization due to the presence of particles.

Observe that it has been assumed that only nonnegative values of S are required to describe the whole Hilbert space of the theory. Indeed the theory is assumed charge-conjugation invariant. Then, negative topological charge states have the same energy and momentum as their charge conjugate states. The assumption that all the states can be described by the NLIE is absolutely not trivial. A mathematical proof of this statement is not yet available, but a number of specific cases supports this conjecture quite strongly.

6.7 The infrared limit of the NLIE and particle scattering

The first task to understand the physics underlying the NLIE we have found, is to identify the theory as a factorized scattering, reconstructing the S-matrix.

In the IR limit $l \to \infty$, the convolution term in (64) vanishes exponentially fast, so it can be dropped. Consider first a state with N_H holes only and XXZ spin $S = N_H/2$.

$$Z(\vartheta) = l \sinh\vartheta + \sum_{j=1}^{N_H} \chi(\vartheta - h_j) \quad , \quad Z(h_j) = 2\pi I_j \tag{69}$$

Observe that the function χ can be written as

$$\chi(\vartheta) = -i \log S_{++}^{++}(\vartheta)$$

where $S_{++}^{++}(\vartheta)$ is the soliton-soliton scattering amplitude in sine-Gordon theory [32], if the parameter $p = \frac{\pi}{\gamma} - 1$ is identified with the one introduced in sG theory $p = \frac{\beta^2}{8\pi - \beta^2}$. This fact is not surprising: eq.(69) can be interpreted physically as a quantization of momenta in a box, for a system of particles interacting with χ as logarithm of the S-matrix. The fact that such quantization is realized supports the interpretation of holes as solitons with

rapidities h_j. This is further evidenced by considering the formula for the energy in such a case. There too the integral term drops at IR, and the formula looks like

$$E = \mathcal{M} \sum_{j=1}^{N_H} \cosh h_j$$

which is the energy of N_H free particles of mass $\mathcal{M}$. The identification with the particular element S_{++}^{++} of the S-matrix forces to give to these solitons a topological charge $+1$ each, which is consistent with the interpretation that $Q = 2S$. To be correct, as $S_{--}^{--} = S_{++}^{++}$ an analogous interpretation is possible in terms of pure antisolitons, all with charge -1, reflecting the charge conjugation invariance of the theory.

Next state to consider is two holes and a complex pair. After some manipulation, the source terms can be arranged, thanks to some identities satisfied by χ functions, in the form

$$Z(\vartheta_i) = l \sinh(\vartheta_i) - i \log S_-(\vartheta_i - \vartheta_j) = 2\pi I_j \ , \ \ i\,,\, j = 1,\, 2$$

where

$$S_-(\vartheta) = -\frac{\sinh\left(\frac{\vartheta + i\pi}{2p}\right)}{\sinh\left(\frac{\vartheta - i\pi}{2p}\right)} S_{++}^{++}(\vartheta)$$

which is the scattering amplitude of a soliton on an antisoliton in the parity-odd channel. The quantum numbers I_+, I_- of the two complex roots are constrained to be $I_\pm = \mp\frac{1}{2}$ for consistency of the IR limit. This state has $S = 0$. It must have topological charge 0, again consistent with $Q = 2S$.

It is known that there are two independent scattering amplitudes for soliton - antisoliton scattering (usually they are presented as transmission and reflection, here the alternative basis of parity even and odd channels is preferred). The even channel is realized by the state with two holes and a selfconjugate root

$$Z(\vartheta_i) = l \sinh(\vartheta_i) - i \log S_+(\vartheta_i - \vartheta_j) = 2\pi I_j \ , \ \ i\,,\, j = 1,\, 2$$

where

$$S_+(\vartheta) = \frac{\cosh\left(\frac{\vartheta + i\pi}{2p}\right)}{\cosh\left(\frac{\vartheta - i\pi}{2p}\right)} S_{++}^{++}(\vartheta)$$

which is the soliton-antisoliton amplitude in the parity-even channel.

Thus, the whole soliton scattering S-matrix of sG theory has been reconstructed. Note that, according to the counting equation (58), in the repulsive regime these 3 possibilities exhaust the possible configurations with $N_H = 2$.

In the attractive regime one has also to consider the breather particles that appear as soliton - antisoliton bound states. It turns out that the breathers are represented by self-conjugate roots (1st breather) or by arrays of wide roots (higher breathers). Consistency of quantization at the IR limit constrains these arrays to appear only when $p < 1$ decreases under the threshold point $p = 1/k$ for the k-th breather. Similar considerations to the above lead to the check that also the breather - soliton and breather - breather S-matrix are correctly reproduced [40, 42].

$N_H = 1, 3$ states have been examined too [41]. In particular the one particle states give the correct description of a single soliton (if $\delta = 1$) or massive Thirring fermion (if $\delta = 0$). In the former case, the particle can be at rest has quantum number $I_h = 0$. In the latter, instead, the lowest possible quantum numbers, corresponding to the lowest possible rapidities, are $I_h = \pm\frac{1}{2}$. There is actually a pair of states, in accordance with the fermion having two components.

6.8 UV limit and vertex operators

This identification gets more support if we consider, for each of the states described above, the opposite $l \to 0$ limit, where we expect to make contact with the UV limit of sG/mTh theory, i.e. with the $c = 1$ CFT described in section 3.7. The UV calculations are usually more difficult to perform than the IR ones, as they involve considering the splitting of the NLIE at UV into two independent left and right parts (as it should be in a good CFT!), the so called *kink equations* and then expressing the energy and momentum in closed form in terms of these, thanks to a lemma presented in [26]. For the details of this sort of manipulations the reader is invited to consult the thesis [30] where all the calculations are done in detail. Here we present the main results and the physical insight they imply.

A first important result is that the $c = 1$ CFT quantum number m (winding number), which is identified with the UV limit of the topological charge, can be related unambiguously to the XXZ spin by $\pm m = 2S$. Of course, the $\pm$ reflects the charge conjugation invariance of the theory. Then, by examining the states we have already visited at IR, we can establish a bridge between particle states and vertex operators of $c = 1$ theory.

1. The **vacuum** state is the one with no holes of complex roots: only the sea of real roots is present. There are two possible choices: $\delta = 0$ or 1 (corresponding, if reinterpreted as $\alpha = 0, \pi$, to periodic and antiperiodic boundary conditions respectively). The result of the UV calculation gives

$$\begin{array}{lll} \text{for } \delta = 0: & \Delta^{\pm} = 0 & \text{i.}e.\ \mathbb{I} \\ \text{for } \delta = 1: & \Delta^{\pm} = \frac{1}{8R^2} & \text{i.}e.\ V_{(\pm 1/2,0)} \end{array}$$

   i.e. the physical vacuum is the one with $\delta = 0$. The other state belongs, with reference to fig. 3, to the unphysical sector IV that never contributes to make a local QFT at UV.

2. The **two-soliton** (two-hole) state, with the minimal rapidity choice obtained with the lowest possible (in absolute value) quantum numbers, gives

(a) for $\delta = 0$ and $I_1 = -I_2 = \frac{1}{2} \Longrightarrow \Delta^{\pm} = \frac{R^2}{2}$ i.*e.* $V_{(0,2)}$.

(b) for $\delta = 1$ and $I_1 = -I_2 = 1 \Longrightarrow$ a $V_{(\pm 1/2,2)}$ descendent, not in UV sG spectrum, as it also belongs to sector IV.

3. The **symmetric soliton-antisoliton** state (two holes and a self-conjugate root), always with the minimal rapidity choice
$\delta = 1$, $I_1 = -I_2 = 1$ and $I_c^{\pm} = 0 \Longrightarrow \Delta^{\pm} = \frac{1}{2R^2}$ i.*e.* $V_{(\pm 1,0)}$

4. The **antisymmetric soliton-antisoliton** state (two holes and a complex pair)
$\delta = 0$, $I_1 = I_c^- = -I_2 = -I_c^+ = \frac{1}{2} \Longrightarrow \Delta^{\pm} = \frac{1}{2R^2}$ i.*e.* $V_{(\pm 1,0)}$
It is obvious that these last two give two linearly independent combinations of the operators $V_{(\pm 1,0)}$, one with even, the other with odd parity.

5. The **one hole** state with $I = 0$, $\delta = 1 \Longrightarrow \Delta^{\pm} = \frac{1}{8R^2}$ i.e. the vertex operator $V_{(0,1)}$, belonging to sector II, thus giving more support to the 1 soliton interpretation. For $\delta = 0$ there are two minimal rapidity states with $I = \pm\frac{1}{2}$. They are identified with the operators $V_{(\pm 1/2,\pm 1)}$. As these states belong to sector III, they are of fermionic nature and actually one identifies them with the components of the Thirring fermion.

6. The **three holes** state with $I_1 = 1$, $I_2 = 0$, $I_3 = -1 \Longrightarrow V_{(0,3)}$. Higher values of quantum numbers give $V_{(n,3)}$, $n \in \mathbb{Z}$

7. The **three holes and close pair** state gives the first nonchiral descendent of $V_{(0,1)}$.

These examples, taken all together, suggest a quantization rule (i.e a consistent choice of δ) to obtain states which are present in the UV spectrum of sine-Gordon or massive Thirring theory. It looks like

$$\begin{array}{cc} Q + \delta + M_{sc} \in 2\mathbb{Z} & \text{for Sine} - \text{Gordon} \\ \delta + M_{sc} \in 2\mathbb{Z} & \text{for massive Thirring} \end{array} \tag{70}$$

where M_{sc} is the number of selfconjugate roots. This selects sector I and II for sine-Gordon states, and sectors I and III for Thirring ones, as it should be from the discussion of sections 3.7 and 4.5. It always excludes sector IV, that does not contain local operators. All this is in accordance with the correct interpretation of Coleman equivalence of Sine-Gordon and Thirring models: even topological charge sectors are identical and the difference of the two models shows up only in the odd topological charge sectors, for which the content of Thirring must be fermionic while that of Sine-Gordon must be bosonic.

To conclude these remarks, we briefly comment about special objects that were introduced in the classification of roots but never used later. Recall their definition: they are roots or holes y_i having $Z'(y_i) < 0$. Now, the function Z is globally monotonically increasing. Indeed its asymptotics for $\vartheta \to \pm\infty$ are dominated by the term $l \sinh \vartheta$ which is obviously monotonically increasing. Also, for l large, this term dominates. Therefore at IR the function Z is surely monotonic and no special objects can appear. This is why we have not considered them in the classification of states above. However, it can happen that

at some critical value l_{crit} of l, coming back from IR towards UV, the convergence of the iterative procedure breaks down, thus revealing that some singularity has been encountered. For the scaling function to be consistently analytically continued after this singularity to reach the UV regime, one needs to introduce modifications to the NLIE creating exactly the contributions that we have called special objects. A more careful analysis reveals that these singularities are produced by the logarithm in the convolution term going off its fundamental branch. A detailed treatment of these objects can be found in [30].

6.9 Twisted NLIE and minimal models

It is a well known fact [43] that the perturbation of the Virasoro minimal model $Vir(r,s)$ by its relevant primary operator $\Phi_{(1,3)}$ is integrable and is described by an RSOS restriction of sine-Gordon theory with

$$p = \frac{r}{s-r} \ . \tag{71}$$

We will use for this model the shorthand notation $Vir(r,s) + \Phi_{(1,3)}$. Al. Zamolodchikov has put forward the idea of modifying sine-Gordon NLIE by a twist α [44] to deal with conformal minimal models. It is to allow this sort of approach that since the beginning we have considered the light-cone lattice construction with a twist ω that induced a twist α in the NLIE. All the physics of sG/mTh model can be done with $\alpha = 0$. Now we explore the situation where α can take well chosen nonzero values implementing the reduction and allowing therefore the NLIE to describe scaling functions of perturbed minimal models too.

Looking first at the ground state, that in analogy with sine-Gordon is expected to be a sea of real roots, the source in NLIE is put to zero and we choose half-integer quantization with $\delta = 0$. The ultraviolet limit of the scaling functions can be computed and gives

$$\tilde{c} = 1 - \frac{6p}{p+1}\left(\frac{\alpha}{\pi}\right)^2 \tag{72}$$

Only in the unitary models $\tilde{c}$ is the Virasoro central charge. Using the rule suggested in (68) for ω and the expression (65) for α gives: $\alpha = \pi/r$. Then

$$\tilde{c} = 1 - \frac{6}{rs} \ ,$$

which is exactly the *effective central charge* $\tilde{c} = c - 24\Delta_{min}$ of the minimal model $Vir(r,s)$. Therefore one can expect that the twisted equation describes the ground state of the model $Vir(r,s) + \Phi_{(1,3)}$. In fact, Fioravanti et al. [38] calculated these scaling functions for the unitary case $s = r+1$ and showed that they match perfectly with the TBA predictions already available. Moreover, choosing the following values for the twist

$$\alpha = \pm\frac{k\pi}{r} \ , \ k = 1 \ldots r-1 \tag{73}$$

they obtained the conformal weights of the operators $\Phi_{(k,k)}$, $k = 1 \dots r-1$ in the UV limit (the sign choice is just a matter of convention). In our notation, $\Phi_{(q,q')}$ denotes the primary field with conformal weights

$$\Delta = \bar{\Delta} = \frac{(qs - q'r)^2 - (s-r)^2}{4sr} \ . \tag{74}$$

The models $Vir(r,s) + \Phi_{(1,3)}$ have exactly $r-1$ ground states. In fact, one can see from the fusion rules that the matrix of the operator $\Phi_{(1,3)}$ is block diagonal with exactly $r-1$ blocks in the Hilbert space made up of states with the same left and right primary weights. In each of these blocks, there is exactly one ground state and for the unitary series $s = r+1$, it was conjectured in [45] that their UV limits are the states corresponding to $\Phi_{(k,k)}$. One can check that in the general non-unitary case the twists (73) correspond in the UV limit to the lowest dimension operators among each of the $r-1$ different blocks of primaries (see explicit examples later).

These ground states are degenerate in infinite volume, but for finite l they split; their gaps decay exponentially as $l \to \infty$. In the unitary case, they were first analyzed in the context of the NLIE in [38] where it was shown that the NLIE predictions perfectly match with the TBA results already available for the unitary series.

Ground states for non-unitary models have been treated in [42]. The simplest example is the scaling Lee-Yang model $Vir(2,5) + \Phi_{(1,3)}$, for which

$$M_B = 2\mathcal{M} \sin \frac{\pi p}{2} = \sqrt{3}\mathcal{M}$$

is the mass of the fundamental particle of the Lee-Yang model (this is more natural here than using the mass $\mathcal{M}$ of the soliton of the unrestricted sine-Gordon model as a scale, since the soliton disappears entirely from the spectrum after RSOS restriction). Call

$$l_B = M_B L = \sqrt{3} l$$

where l is the variable appearing in NLIE. There is only one independent value of the twist

$$\alpha = \frac{\pi}{2} \ .$$

There is only one ground state in this model, which corresponds to the primary field with conformal weights

$$\Delta = \bar{\Delta} = -\frac{1}{5} \ ,$$

All the models of the class $Vir(2, 2n+1) + \Phi_{(1,3)}$ have only one ground state. For models with two ground states, we can take a look at $Vir(3,5)$ ($Vir(3,7)$ was also taken into account in [42]). For $Vir(3,5)$ the ultraviolet spectrum is defined by the following Kac table, where the weight of the field $\Phi_{(k,l)}$ is found in the k-th row and l-th column.

0	$-1/20$	$1/5$	$3/4$
$3/4$	$1/5$	$-1/20$	0

The two blocks of the perturbing operator $\Phi_{(1,3)}$ are defined by the fields $\{\Phi_{(1,2)}\,,\,\Phi_{(1,4)}\}$ and $\{\Phi_{(1,1)}\,,\,\Phi_{(1,3)}\}$, respectively. The ground states correspond in the UV to the operators $\Phi_{(1,2)}$ and $\Phi_{(1,1)}$, as can be checked directly using formulae (72), (71) and (73).

Now we turn our attention to excited states over these minimal model vacua. We restrict ourselves to the case of neutral (i.e. $S = 0$) states. Even for states with a zero charge the relation between α and ω is highly nontrivial.

It is quite easy to show that choosing the value of ω as

$$\omega = \frac{k\pi}{p+1}$$

where k is integer, we can reproduce all the required values of α listed in equation (73). The twisted lattice Bethe Ansatz was analyzed by de Vega and Giacomini in [46]. On the lattice, passing from the sine-Gordon model to the perturbed Virasoro model amounts to going from the six-vertex model to a lattice RSOS model. In [46] it was shown that to obtain all the states of the RSOS model it is necessary to take all the twists

$$\omega = \frac{k\pi}{p+1} \mod \pi$$

into account. The fact that not all these twists correspond to inequivalent values of α and so to different physical states is a consequence of the RSOS truncation.

We remark that the parameter α drops out of the second determination of Z in the attractive regime. This is important because as a consequence the IR asymptotics of the breather states do not depend on α and so the S-matrices involving breathers are unchanged. In fact, scattering amplitudes between solitons and breathers remain unchanged too. This matches with the fact that the RSOS restriction from sine-Gordon theory to perturbed minimal models does not modify scattering amplitudes that involve two breathers or a breather and soliton [43].

Let us start by examining the scaling Lee-Yang model $Vir(2,5) + \Phi_{(1,3)}$. Since we have a single ground state, there are no kinks in the spectrum. We fix the value of α as above, so we still have a freedom of choosing δ. This can be done by matching to the UV dimensions: if for a certain state we choose the wrong value of δ, we find a conformal dimension that is not present in the Kac table of the model.

The excited states are multi-particle states of the first breather of the corresponding unrestricted sine-Gordon model, which has $p = \frac{2}{3}$. One can calculate the state containing one particle at rest. It turns out that as we decrease l, the self-conjugate root starts moving to the right. It does not remain in the middle like in the $\alpha = 0$ case, which is to be expected since for nonzero α we have no left/right symmetry. However, the total momentum of the state still remains zero due to a contribution from the integral term in momentum equation (67). We have here an example of the phenomenon of the appearance of the special root and its two accompanying holes: the numerical iteration breaks down

at around $l_{crit} = 2.5$. For $l < l_{crit}$ the NLIE must be corrected by the introduction of this special root and the two accompanying holes (which are not special). They must be taken into account to compute the UV limit correctly. Notice that the two holes thus introduced do not have any counterpart at IR and it is not possible to give them any interpretation as particles. Their quantum numbers cannot be freely chosen. They are constrained to take the same value of the quantum number of the special root, which in turn is fixed by the constraints of the problem.

We know that the Lee-Yang model contains only two primary fields, the identity $\mathbb{I}$ and the field φ with left/right conformal weights $-1/5$. In fact, the ground state of the massive model corresponds to φ in the UV limit. For the one particle state, it turns out that the special root and one of the holes move to the left together with the self-conjugate root, while the other hole moves to the right. This gives, after the "kink" NLIE calculations are done, $\Delta = \bar{\Delta} = 0$, i.e. the identity operator $\mathbb{I}$, as expected from TCSA in [33].

Let us look now at moving breathers. If the self-conjugate root has Bethe quantum number $I = 1$, the corresponding state will have momentum quantum number 1, i.e.

$$P = \frac{2\pi}{R} \ ,$$

and in the UV $\Delta - \bar{\Delta} = 1$. No special root appears here. The reason is that the self-conjugate root moves to the left and the real part of its position ϑ is given to leading order by

$$\sinh(\Re e\, \vartheta) \ \sim \ -\frac{2\pi I}{l_B} \ .$$

As a result, the contribution to the derivative of Z from the $l \sinh\vartheta$ term remains finite when $l \to 0$. In the previous example of the particle at rest the left-moving nature of the self-conjugate root when $I = 0$ does not prevent the occurrence of the breakdown in the iteration scheme: since its Bethe quantum number is zero, it does not move fast enough to the left in order to balance the negative contribution to derivative of Z coming from the self-conjugate root source. At the moment we have no way of predicting analytically whether or not there will be specials in the UV limit: we just use the numerical results to establish the configuration for the evaluation of UV weights, supplemented with a study of the self-consistency of the solution of the kink equation. The UV dimensions for the moving breather turn out to correspond to the state $L_{-1}\varphi$.

One can similarly compute the UV dimensions for some other excited states. For example, the two-particle states with half-integer Bethe quantum numbers $I_1 > 0,\ I_2 < 0$ for the two self-conjugate roots are found to have

$$\Delta = -\frac{1}{5} + I_1 + \frac{1}{2} \ , \ \bar{\Delta} = -\frac{1}{5} - I_2 + \frac{1}{2} \ ,$$

which shows that they correspond in the UV to descendent states of φ. The first such state with quantum numbers

$$I_1 = \frac{1}{2} \ , \ I_2 = -\frac{1}{2}$$

corresponds to $L_{-1}\bar{L}_{-1}\varphi$.

The lowest lying three-particle state of zero momentum, with Bethe quantum numbers $(-1,0,1)$ corresponds to the left/right symmetric second descendent of the identity field, i.e. to the field $T\bar{T}$, where T denotes the energy-momentum tensor. This is very interesting, since from experience with NLIE UV calculations one would naively expect this to be a first descendent (descendent numbers are usually linked to the sum of Bethe quantum numbers of left/right moving particles and this state is the lowest possible descendent of the identity $\mathbb{I}$). However, the field $L_{-1}\bar{L}_{-1}\mathbb{I}$ is well-known to be a null field in any conformal field theory. It seems from this example that NLIE is clever enough to avoid null vectors in minimal models, but of course more investigation is needed in this respect.

The above correspondences are confirmed by comparing to TCSA data (see the wonderful figures in [33]). In general, one can establish the rule that states with odd number of particles must be quantized by integers ($\delta = 1$), while those containing even number of particles must be quantized by half-integers ($\delta = 0$) in order to reproduce correctly the spectrum of the scaling Lee-Yang model.

We conducted similar studies for the models $Vir(2,7)+\Phi_{(1,3)}$ and $Vir(2,9)+\Phi_{(1,3)}$ and found similarly good agreement with TCSA data. For the first one-particle state of the model $Vir(2,7)+\Phi_{(1,3)}$ we also checked our results against the TBA data in the numerical tables of [36] and found agreement with the TBA results.

Given the choice of α above, the correct rule of quantization in all of the models $Vir(2,2n+1)+\Phi_{(1,3)}$ is

$$\delta + M_{sc} \in 2\mathbb{Z}$$

where M_{sc} is the number of self-conjugate roots in the source corresponding to the state. This is exactly the same rule as the one established for pure sine-Gordon theory in [28]. In the presence of the twist, such a rule of course has meaning only together with a definite convention for the choice of α.

It is interesting to note that in the case of $Vir(3,n)$ models, all the neutral states must come in two copies, since they can be built on top of either of the two ground states. We take the example of the $Vir(3,7)$ model and the states corresponding to a breather at rest. We have $p = \frac{3}{4}$, but now there are two inequivalent values for the twist

$$\alpha = \frac{\pi}{3}\ ,\ \frac{2\pi}{3}\,.$$

When $\alpha = 0$, we can estimate the critical value of l where specials appear to be $l_{crit} = 5.23$. In this case, the twist helps a bit, because it makes the self-conjugate root a left mover; it is intuitively clear that the bigger the twist, the more it lowers the eventual value of l_{crit}. From TCSA data one can identify that breather #1 is really the one-particle state in the sector of ground state #1 ($\alpha = \pi/3$), while breather #2 is in the sector over ground state #2 ($\alpha = 2\pi/3$).

A direct calculation of the conformal weights gives $\frac{3}{28}$ for breather #1 and 0 for breather #2, which are in complete agreement with TCSA data. We also checked the two different

states containing two breathers with Bethe quantum numbers $I_1 = \frac{1}{2}$, $I_2 = -\frac{1}{2}$, and found an equally excellent numerical agreement with TCSA. Just like in the case of sine-Gordon and scaling Lee-Yang model, for these states one can continue the iteration of the NLIE down to any small value of l, although at the expense of a growing number of necessary iterations to achieve the prescribed precision.

7 Conclusions

We have shown how the nonlinear integral equation deduced from the light cone lattice model of [22] describes exactly the Finite Size Effects of the sine-Gordon/massive Thirring theory, taking into account excited states too. The most important results are summarized as follows:

1. By examining the infrared limits of the equation it has been shown that it leads to the correct two-particle S-matrices for both scattering states and bound states;

2. By computing the UV conformal weights from the NLIE we have shown that the energy-momentum spectrum is consistent with the UV one of sG/mTh theory only if we choose the parameter δ (i.e. the quantization rule) as indicated in (70).

3. The predictions of the NLIE have been verified by comparing them to results coming from the TCSA approach (for sG/mTh).

4. The framework required to deal with minimal models perturbed by $\Phi_{(1,3)}$ is built up. Many examples and numerical/analytical checks are given for the ground state in the unitary [38] and non unitary cases.

5. All the conformal dimensions can be reproduced (Kac table) with a convenient choice of the twist (73). The particular relation suggested in [47] is not enough to describe the whole spectrum.

6. The IR computations reproduce correctly the S-matrix of perturbed minimal models (at list in the cases not involving kinks), because the $p < 1$ second determination drops the twist.

7. Numerical calculations (TCSA) of concrete examples and comparison with TBA where possible, give a strong evidence for the correctness of the energy levels derived from the twisted NLIE for excited states, both for the full sG/mTh model and for the Restricted models.

The understanding of "(twisted) sine-Gordon NLIE" and of the finite size behaviour of the continuum theory defined from the NLIE (64) is not quite complete. Indeed some open questions have not an answer yet and further interesting developments may be forecast. We close this review by pointing to possible interesting questions that need further investigation.

1. Is the set of scaling functions provided by the NLIE complete i.e. can we find to every sG/mTh or minimal model state a solution of the NLIE describing its finite volume behaviour? The main difficulty in this "counting problem" is that the structure of the solutions is highly dependent on the value of the coupling constant – consider e.g. the appearance of special sources.

2. The multi-kink states characteristic of the perturbed minimal models (except for the series $Vir(2, 2n+1)$) have been omitted in the analysis performed in the last section. Although the general treatment is valid for such states too, a detailed description is far more complicated than for states which contain only breathers and is left open to further studies.

3. There is also an unresolved technical difficulty, namely that the source configuration of the NLIE may change as we vary the volume parameter l. Typically what happens is that while the counting function Z is monotonic on the real axis for large volume, this may change as we lower the value of l and so-called special sources (and accompanying holes) may appear. We do not as yet have any consistent and tractable numerical iteration scheme to handle this situation, although the analytic UV calculations and intuitive arguments show that the appearance of these terms in the NLIE is consistent with all expectations coming from the known properties of perturbed CFT. In addition, in the range of l where we can iterate the NLIE without difficulty, our numerical results show perfect agreement with TCSA. We want to emphasize that these transitions are not physical: the counting function Z and the energy of the state are expected to vary analytically with the volume. It is just their description by the NLIE with requires a modification of source terms. As it was pointed out also in [27, 28], the whole issue is related to the choice of the branch of the logarithmic term in the NLIE (64).

4. The problem pointed out at the previous point is very similar to the behaviour of singularities encountered in the study of the analytic continuation of the TBA equation [36] and we can hope that establishing a closer link between the two approaches can help to clarify the situation. From the form of the source terms in the NLIE it seems likely that the excited state equations can be obtained by an analytic continuation procedure analogous to the one used in TBA [36] to obtain the excited state TBA equations. Certain features of the arrangement of the complex roots in the attractive regime and their behaviour at breather thresholds also point into this direction. This is an interesting question to investigate, because it can shed light on the organization of the space of states and can lead closer to solving the counting problem described above.

5. Even if on the lattice the Bethe vectors given in (3) completely describe the Hilbert space of the XXZ chain, the form of eigenvectors for continuum energy and momentum is completely unknown. This fact reflects the unresolved question of the form of energy and momentum eigenvectors for the minkowskian sine-Gordon theory (the

Faddeev-Zamolodchikov algebra is only a phenomenological picture). Also the determination of correlation functions is in an early stage, although in recent publications [48] explicit expressions for the field $\langle e^{ia\varphi}\rangle$ and its descendent are given.

6. The extension of light-cone approach and NLIE to other QFT models is in progress. An interesting extension is the description of finite volume spectrum of Zhiber-Mikhailov-Shabat model (imaginary Bullough-Dodd). A first suggestion in this direction, even if is obtained in a completely different approach, recently appeared in the [49], for the ground state. Early attempts to get the excited states can be found in [50]. Extensions to vertex models based on higher rank simply-laced Lie Algebras have also been investigated [51, 47] and should correspond – a detailed analysis has not been done yet – to the solitonic (imaginary coupling) affine Toda Field Theories.

The second and fourth points are important because their investigation may lead closer to understanding the relation between the TBA and the NLIE approaches. It is quite likely that establishing a connection between the two methods would facilitate the development of both and may point to some common underlying structure.

As a final remark, let us mention an application of the NLIE to a chemical compound, i.e. "copper Benzoate" ($Cu(C_6D_5COO)_2{\cdot}3D_2O$). Its specific heat has been computed in [19] using the vacuum untwisted ($\alpha = 0$) NLIE, where the finite size L is the inverse temperature $L = (k_B T)^{-1}$. This corresponds to do equilibrium thermodynamics of sine-Gordon model. It is curious to notice that in that context the NLIE is called Thermal Bethe Ansatz, emphasizing the well known relation between statistical mechanics and QFT. Exactly in a similar contest (Heisenberg ferromagnet model) the original Bethe Ansatz [37] was born.

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References

[1] P. Christe, M. Henkel, "Introduction to Conformal invariance and its Applications to Critical Phenomena", Springer (1993), Berlin

[2] J. L. Cardy, *Nucl. Phys.* **B270** (1986) 186

[3] M. Lüscher, *Comm. Math. Phys.* **104** (1986) 177

[4] M. Lüscher, *Comm. Math. Phys.* **105** (1986) 153

[5] K. G. Wilson and J. Kogut, *Phys. Rev.* **C12** (1974) 200

[6] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, 1976)

[7] C. G. Callan, S. Coleman and R. Jackiw, *Ann. Phys.* **59** (1970) 42

[8] A. B. Zamolodchikov, *Sov. J. Nucl. Phys.* **46 (6)** (1987) 1090

[9] A. B. Zamolodchikov, *JETP Lett.* **43** (1986) 730

[10] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333

[11] D.Friedan, Z.Qiu and S.Shenker, *Phys. Rev. Lett.* **52** (1984) 1575; in *Vertex operators in Mathematics and Physics*, J.Lepowsky, S.Mandelstam and I.Singer eds. (Springer, Berlin 1985); *Comm. Math. Phys.* **107** (1986) 535

[12] Vl.S.Dotsenko and V.A.Fateev, *Nucl. Phys.* **B240** (1984) 312; *Nucl. Phys.* **B251** (1985) 691

[13] T. Klassen and E. Melzer, *Int. J. Mod. Phys.* **A8** (1993) 4131

[14] A. B. Zamolodchikov, *Adv. in Pure Math.* **19**

[15] A. Cappelli and J. I. Latorre, *Nucl. Phys.* **B340** (1990) 659

[16] G. Feverati, E. Quattrini and F. Ravanini, *Phys. Lett.* **B374** (1996) 64

[17] T. Klassen and E. Melzer, *Nucl. Phys.* **B350** (1991) 635

[18] A. W. W. Ludwig and J. L. Cardy, *Nucl. Phys.* **B285** (1987) 687

[19] F. H. L. Essler, *Sine-Gordon low-energy effective theory for Copper Benzoate*, to appear in *Phys. Rev. B,* cond-mat/9811309

[20] S. Coleman, *Phys. Rev.* **D11** (1975) 2088

[21] S. Mandelstam, *Phys. Rev.* **D11** (1975) 3026

[22] C. Destri and H.J. de Vega, *Nucl. Phys.* **B290** (1987) 363

[23] R. Baxter, *Exactly solved models in statistical mechanics*, Academic Press (London)

[24] L.D. Faddeev and L.A. Takhtadzhyan, *Russ. Math. Surv.* **34** (1979) 11

[25] H.J. de Vega, *Int. Jour. Mod. Phys.* **A4** (1989) 2371

[26] C. Destri and H.J. de Vega, *Phys. Rev. Lett.* **69** (1992) 2313; *Nucl. Phys.* **B438** (1995) 413

[27] C. Destri and H.J. de Vega, *Nucl. Phys.* **B504** (1997) 621

[28] G. Feverati, F. Ravanini and G. Takács, *Nucl. Phys.* **B540** (1999) 543

[29] L. D. Faddeev in *Les Houches Summer Course Proceedings* (1995), hep-th 9605187

[30] G. Feverati, *Finite Volume Spectrum of Sine-Gordon Model and its Restrictions,* Ph.D. Thesis, Bologna University (2000), hep-th/0001172

[31] C. Destri and H.J. de Vega, *J. Phys.* **A22** (1989) 1229

[32] A.B. and Al.B. Zamolodchikov, *Ann. of Phys.* **120** (1979)

[33] V. P. Yurov and A. B. Zamolodchikov, *Int. J. Mod. Phys.* **A5** (1990) 3221

[34] Al. B. Zamolodchikov, *Nucl. Phys.* **B342** (1990) 695

[35] A. Klümper and P.A. Pearce, *J. Stat. Phys.* **64** (1991) 13
A. Klümper, M. Batchelor and P.A. Pearce, *J. Phys.* **A24** (1991) 3111

[36] P. Dorey and R. Tateo, *Nucl. Phys.* **B482** (1996) 639
P. Dorey and R. Tateo, *Nucl. Phys.* **B515** (1998) 575

[37] H. Bethe, *Z. Phys.* **71** (1931) 205

[38] D. Fioravanti, A. Mariottini, E. Quattrini and F. Ravanini, *Phys. Lett.* **B390** (1997) 243

[39] G. Feverati, F. Ravanini and G. Takács, *Phys. Lett.* **B430** (1998) 264

[40] A. Tani, *Equazioni Integrali Non-Lineari e Funzioni di Scala in Teorie di Campo Integrabili,* Laurea thesis (in Italian), Bologna University (1999), available at http://www-th.bo.infn.it/hepth/papers.html

[41] G. Feverati, F. Ravanini and G. Takács, *Phys. Lett.* **B444** (1998) 442

[42] G. Feverati, F. Ravanini and G. Takács, *Nucl. Phys.* **B570** (2000) 615

[43] N. Reshetikhin and F. Smirnov, *Comm. Math. Phys.* **131** (1990) 157-181

[44] Al. B. Zamolodchikov, *Nucl. Phys.* **B432** (1994) 427-456
Al. B. Zamolodchikov, *Phys. Lett.* **B335** (1994) 436-443

[45] T. Klassen and E. Melzer, *Nucl.Phys.* **B370** (1992) 511

[46] H. J. de Vega and H. J. Giacomini, *J. Phys.* **A22** (1989) 2759

[47] P. Zinn-Justin, *J. Phys.* **A31** (1998) 6747

[48] S. Lukyanov and A. Zamolodchikov, *Nucl.Phys.* **B493** (1997) 571
V. Fateev, D. Fradkin, S. Lukyanov, A. Zamolodchikov and Al. Zamolodchikov, *Nucl. Phys.* **B540** (1999) 587

[49] P. Dorey and R.Tateo, *Nucl. Phys.* **B571** (2000) 583

[50] M. Bellacosa, *Equazione integrale non lineare e spettro di volume finito per teorie di Toda affini* $A_2^{(2)}$*,* Laurea thesis (in Italian), Bologna University (2000), available at http:/www-th.bo.infn.it/hepth/papers.html

[51] A. Mariottini, *Ansatz di Bethe Termodinamico ed Equazione di Destri-de Vega in Teorie di Campo Bidimensionali,* Laurea thesis (in Italian), Bologna University (1996), available at http://www-th.bo.infn.it/hepth/papers.html